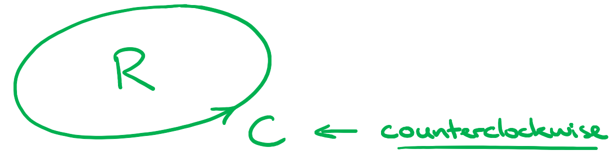
13. The Cauchy-Goursat Theorem

Green's Theorem:



$P(x,y)$, $Q(x,y)$ real valued functions on R and C with continuous first derivatives.
"C^1 functions"

$$\boxed{\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dx\,dy.}$$

Application of Green's thm. to analytic functions

$f(z) = u(x,y) + iv(x,y)$ analytic on C and R with continuous derivatives.

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+i\,dy)$$

$$= \int_C u\,dx - v\,dy \;+\; i\int_C v\,dx + u\,dy$$

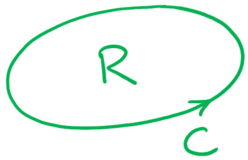
$$\underset{\substack{\uparrow \\ \text{Green's} \\ \text{theorem}}}{=} \iint_R \underbrace{(-v_x - u_y)}_{\substack{=0 \text{ by the} \\ \text{C.-R. eqn.s}}}\,dx\,dy + i\iint_R \underbrace{(u_x - v_y)}_{\substack{=0 \text{ by the} \\ \text{C.-R. eqn.s}}}\,dx\,dy$$

$\rightsquigarrow$ $\displaystyle\int_C f(z)\,dz = 0$ provided

(1) C is a simple closed contour;

Needed to justify using Green's thm. $\longrightarrow$ (2) $f'(z)$ is continuous.

Cauchy-Goursat Theorem:



If a function $f(z)$ is analytic at all points interior to and on a simple closed contour C, then

$$\int_C f(z)\,dz = 0.$$

Remarks:

① The Cauchy-Goursat theorem follows from Green's theorem provided we assume that $f'(z)$ is continuous. But we really don't want to assume this, since we are eventually going to use the Cauchy-Goursat theorem to prove that analytic functions have continuous derivative (indeed we will prove that analytic functions have analytic derivative, and so are infinitely diff$^{\text{ble}}$). It turns out that we do not need to assume that $f'(z)$ is continuous! This was shown by Goursat.

② The proof (based on Goursat's idea) can be found in the book. Pages 150-154 in Brown & Churchill, 9^{th} ed.

Optional! Highly recommended for pure math majors.

③ For a nicer (and more standard) proof, see Stein & Shakarchi's book "Complex Analysis".

④ For a readable proof of the Jordan Curve Theorem, see Fulton's introductory textbook on "Algebraic Topology". This book gives a very gentle introduction to this subject, and complements our course extremely well, so is well worth taking a look at.

Example: $\int_C e^{z^2} \cos(z^5)\, dz = 0$

for any closed contour C in $\mathbb{C}$,

since $f(z) = e^{z^2}\cos(z^5)$ is entire.

analytic in all of $\mathbb{C}$.

Let $D \subseteq \mathbb{C}$ be a domain.

Def^n: D is a simply connected domain

$\Updownarrow$

every simple closed contour in D encloses only points in D.

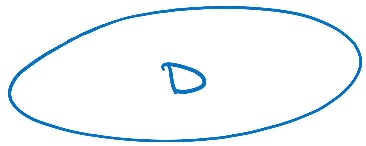
"D has no holes"

Examples:

① open unit disc $D = \{|z|<1\}$ ✓

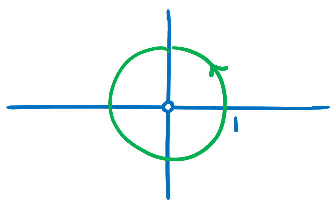
② $D = \mathbb{C}$ ✓

③ D any convex domain. ✓



④ $D = \mathbb{C} - \{0\}$ "punctured plane"

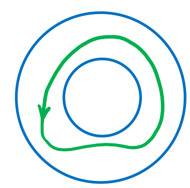
$\mathbb{C} - \{0\}$ is not simply connected!



The unit circle is in D, and encloses $0 \notin D$.

⑤ annulus $D = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

D is not simply connected!



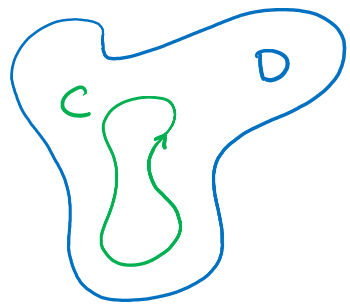
The following is a consequence of the Cauchy-Goursat theorem:

Theorem: If C is a closed contour in a simply connected domain D and $f(z)$ is analytic in D then

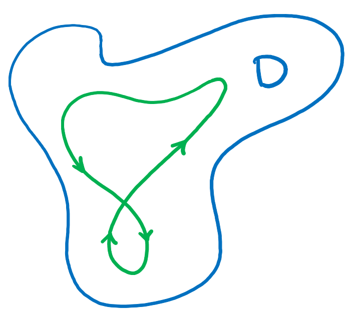
$$\int_C f(z)\,dz = 0.$$

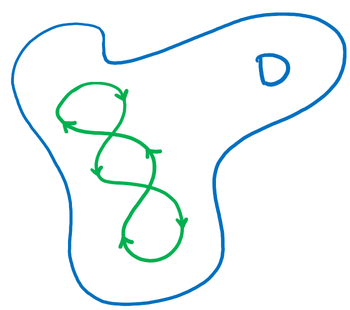
Proof:

If C is simple, apply Cauchy-Goursat.



If C self-intersects once, then split C into 2 simple closed contours and apply Cauchy-Goursat. Similarly if C has multiple self-intersections...





$\square$

Since path independence of contour integrals guarantees the existence of an antiderivative, we have as a corollary:

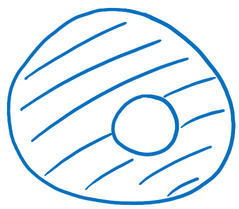
Theorem: Any analytic function in a simply connected domain D admits an antiderivative in D.

Simply vs Multiply Connected Domains

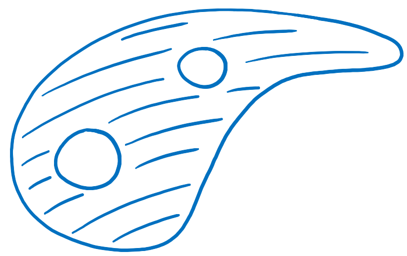
Simply connected domains have "no holes" inside.

Domains that are not simply connected are called multiply connected.

"has holes"



1 hole



2 holes

Example: $D = \{0 < |z| < 1\}$

has "1 hole" consisting of a single point 0.

* Mention homotopy and homology.