15. The Cauchy Integral Formula for Derivatives

The Cauchy Integral Formula tells us that we can compute the value of an analytic function $f$ at a point inside a simple closed contour $C$ from its values on $C$, by an integral.

It turns out that we can also compute the values of the derivatives of $f$ by integrating. In particular, we can show that an analytic function (a function which is complex different on an open set) is in fact infinitely (complex) differentiable.

**Cauchy Integral Formula:**

$r_C$ simple closed contour, pos. oriented;
$f$ analytic on $C$ and inside $C$;
$z$ any point interior to $C$.

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(s)}{s-z} \, ds.$$
Lemma: \( (*) \) can be differentiated with respect to \( z \), giving

\[
\frac{d}{dz} \left( \frac{f(s)}{s-z} \right) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z)^2} \, ds.
\]

To see this we differentiate \( \int_C \frac{f(s)}{s-z} \, ds \) under the integral sign (using \( \frac{d}{dz} \left[ \frac{1}{s-z} \right] = \frac{1}{(s-z)^2} \)) to get \( \int_C \frac{f(s)}{(s-z)^2} \, ds \). This can be justified using some facts from real analysis.

What about higher derivatives? No problem!

\[
\frac{d^2}{dz^2} \left( \frac{1}{s-z} \right) = \frac{2}{(s-z)^3}
\]

\[
\frac{d^3}{dz^3} \left( \frac{1}{s-z} \right) = \frac{2 \times 3}{(s-z)^4}
\]

\[\vdots\]

\[
\frac{d^n}{dz^n} \left( \frac{1}{s-z} \right) = \frac{n!}{(s-z)^{n+1}}, \quad n = 1, 2, 3, \ldots
\]

\[
\Rightarrow \frac{d^n}{dz^n} \int_C \frac{f(s)}{s-z} \, ds = n! \int_C \frac{f(s)}{(s-z)^{n+1}} \, ds.
\]
Theorem: If \( f \) and \( C \) are as in (*)
then \( f \) is infinitely complex differentiable and

\[
f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(s)}{(s-z)^{n+1}} \, ds \quad (n=1,2,3,...)
\]

for all points \( z \) interior to \( C \).

Remarks:
1. The theorem is also valid for \( n=0 \)
   (recall that \( 0! = 1 \)), in which case one
   just gets back the usual Cauchy Integral
   Formula.
2. It follows that if
   \( f(z) = u(x,y) + iv(x,y) \) is an analytic
   function, then \( u \) \& \( v \) have continuous
   derivatives (in fact they are infinitely differentiable).
   This eliminates the gap in our proof that
   \( u \) \& \( v \) are harmonic.
3. One application of the theorem is
   to compute integrals, by writing the formula as

\[
\oint_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz = 2\pi i \, \frac{f^{(n)}(z_0)}{n!}.
\]
Examples:

1. Take \( f(z) = 1 \).

\[
\int_C \frac{dz}{(z-z_0)^{n+1}} = 2\pi i \frac{f^{(n)}(z_0)}{n!} = \begin{cases} 
2\pi i, & n = 0 \\
0, & n = 1, 2, 3, \ldots
\end{cases}
\]

for any simple closed contour, positively oriented, that encloses \( z_0 \).

2. Evaluate \( \int_C \frac{z^2}{(z-z_0)^{n+1}} \, dz \), \( C \) as above.

\[
= \frac{2\pi i}{n!} f^{(n)}(z_0)
\]

where \( f(z) = z^2 \)

\[
\int_C \frac{z^2}{(z-z_0)^{n+1}} \, dz = \begin{cases} 
2\pi i z_0^2, & n = 0 \\
4\pi i z_0, & n = 1 \\
2\pi i, & n = 2 \\
0, & n \geq 3
\end{cases}
\]

3. Evaluate \( \int_C \frac{e^{3z}}{z^2} \, dz \) where \( C \) is the unit circle, taken counterclockwise.

\( \Rightarrow \) take \( f(z) = e^{3z} \), \( z_0 = 0 \), \( n = 1 \),

\& use \( \int_C \frac{f(z)}{(z-z_0)^{n+1}} \, dz = 2\pi i \frac{f^{(n)}(z_0)}{n!} \).

\( f'(z) = 3e^{3z} \), \( f'(0) = 3 \).
\[ \int_{C} \frac{e^{2z}}{z^2} \, dz = 6\pi i. \]

**Cauchy's Inequalities:**

\( z_0 \in \mathbb{C}, \quad R > 0, \quad C_R \) circle of radius \( R \) centered at \( z_0 \), taken counterclockwise.

\[ f(z) \text{ analytic inside and on } C_R. \]

\[ |f(z)| \leq M_R \quad \text{for all } z \in C_R. \]

\[ \implies |f^{(n)}(z_0)| \leq \frac{n! M_R}{R^n} \quad \text{for } n = 0, 1, 2, 3, \ldots \]

**Proof:**

\[ f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}} \, dz \]

\[ \implies |f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \cdot \frac{M_R}{R^{n+1}} \cdot \frac{2\pi R}{\text{Length}(C_R)} \]

\[ \implies |f^{(n)}(z_0)| \leq \frac{n! M_R}{R^n} \]