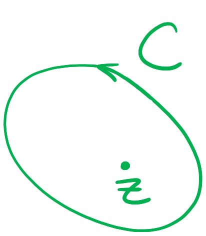
15. The Cauchy Integral Formula for Derivatives

The Cauchy Integral Formula tells us that we can compute the value of an analytic function f at a point inside a simple closed contour C from its values on C, by an integral.

It turns out that we can also compute the values of the derivatives of f by integrating. In particular, we can show that an analytic function (a function which is complex diff^ble on an open set) is in fact infinitely (complex) diff^ble.

Cauchy Integral Formula:



C simple closed contour, pos. oriented;

f analytic on C and inside C;

z any point interior to C.

$$\rightsquigarrow \quad f(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{s-z}\,ds. \qquad (*)$$

Lemma: (*) can be differentiated with respect to z, giving

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\,ds.$$

To see this we differentiate $\int_C \frac{f(s)}{s-z}\,dz$ under the integral sign (using $\frac{d}{dz}\left[\frac{1}{s-z}\right] = \frac{1}{(s-z)^2}$) to get $\int_C \frac{f(s)}{(s-z)^2}\,dz$. This can be justified using some facts from real analysis.

What about higher derivatives? No problem!

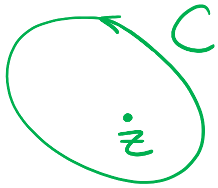
$$\frac{d^2}{dz^2}\left(\frac{1}{s-z}\right) = \frac{2}{(s-z)^3}$$

$$\frac{d^3}{dz^3}\left(\frac{1}{s-z}\right) = \frac{2\times 3}{(s-z)^4}$$

$$\vdots$$

$$\frac{d^n}{dz^n}\left(\frac{1}{s-z}\right) = \frac{n!}{(s-z)^{n+1}}, \quad n = 1,2,3,\ldots$$

$$\rightsquigarrow \frac{d^n}{dz^n}\int_C \frac{f(s)}{s-z}\,ds = n!\int_C \frac{f(s)}{(s-z)^{n+1}}\,ds.$$



Theorem: If f and C are as in $(*)$, then f is infinitely complex diff^{ble} and

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(s)}{(s-z)^{n+1}}\,ds \qquad (n=1,2,3,\ldots)$$

for all points z interior to C.

Remarks:

① The theorem is also valid for $n=0$ (recall that $0! = 1$), in which case one just gets back the usual Cauchy Integral Formula.

② It follows that if $f(z) = u(x,y) + i\,v(x,y)$ is an analytic function, then u & v have continuous derivatives (in fact they are infinitely diff^{ble}). This eliminates the gap in our proof that u & v are harmonic.

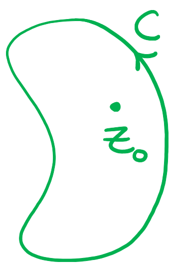
③ One application of the theorem is to compute integrals, by writing the formula as

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = 2\pi i\,\frac{f^{(n)}(z_0)}{n!}.$$

Examples:

① Take $f(z) = 1$.

$$\int_C \frac{dz}{(z-z_0)^{n+1}} = 2\pi i \, \frac{f^{(n)}(z_0)}{n!} = \begin{cases} 2\pi i, & n = 0 \\ 0, & n = 1, 2, 3, \ldots \end{cases}$$



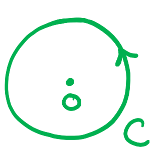
for any simple closed contour, positively oriented, that encloses z_0.

② Evaluate $\underbrace{\int_C \frac{z^2}{(z-z_0)^{n+1}}\,dz}_{= \frac{2\pi i}{n!} f^{(n)}(z_0)}$, C as above.

where $f(z) = z^2$

$$\int_C \frac{z^2}{(z-z_0)^{n+1}}\,dz = \begin{cases} 2\pi i \, z_0^2, & n = 0 \\ 4\pi i \, z_0, & n = 1 \\ 2\pi i, & n = 2 \\ 0, & n \geq 3. \end{cases}$$

③ Evaluate $\int_C \frac{e^{3z}}{z^2}\,dz$ where C is the unit circle, taken counterclockwise.



$\rightsquigarrow$ take $f(z) = e^{3z}$, $z_0 = 0$, $n = 1$,

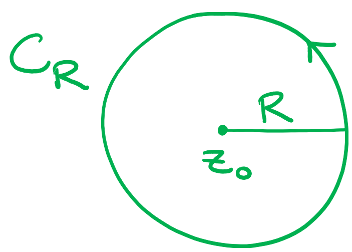
& use $\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = 2\pi i \, \frac{f^{(n)}(z_0)}{n!}$.

$f'(z) = 3e^{3z}$, $f'(0) = 3$.

$$\rightsquigarrow \quad \int_C \frac{e^{3z}}{z^2}\, dz = 6\pi i .$$

Cauchy's Inequalities:

$z_0 \in \mathbb{C}$, $R > 0$, C_R circle of radius R centered at z_0, taken counterclockwise.



$f(z)$ analytic inside and on C_R.

$|f(z)| \leq M_R$ for all $z \in C_R$.

$$\Longrightarrow \quad \boxed{\left| f^{(n)}(z_0) \right| \leq \frac{n!\, M_R}{R^n}} \qquad n = 0, 1, 2, 3, \ldots$$

Proof:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_R} \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

$$\rightsquigarrow \quad \left| f^{(n)}(z_0) \right| \leq \frac{n!}{2\pi} \cdot \underbrace{\frac{M_R}{R^{n+1}}}_{\substack{\Uparrow \\ \text{bound for} \\ \text{the integrand}}} \cdot \underbrace{2\pi R}_{= \text{Length}(C_R)}$$

$$\rightsquigarrow \quad \left| f^{(n)}(z_0) \right| \leq \frac{n!\, M_R}{R^n} \qquad \square$$