Page 71

4(a) Let \( z = re^{i\theta} \). Then, \( f(z) = \frac{1}{z^4} = \frac{1}{r^4} e^{-4i\theta} = \frac{1}{r^4} \cos(4\theta) - i\frac{1}{r^4} \sin(4\theta) = u(r, \theta) + iv(r, \theta) \). Then, observe that \( u_r = -4r^{-5}\cos(4\theta) \), \( u_\theta = -4r^{-4}\sin(4\theta) \) and \( v_r = 4r^{-5}\sin(4\theta) \), \( v_\theta = -4r^{-4}\cos(4\theta) \). Now clearly all partial derivatives are continuous and it’s easy to check that \( ru_r = v_\theta \) and \( u_\theta = -rv_r \) for all \( \theta \) and \( r \neq 0 \). Therefore, \( f \) is differentiable everywhere in the domain and \( f'(z) = e^{-i\theta} (u_r + iv_r) = e^{-i\theta} (-4r^{-5}\cos(4\theta) + 4r^{-5}i\sin(4\theta)) = -4r^{-5} e^{-i\theta} e^{-4i\theta} = -\frac{4}{r} e^{-i\theta} = -\frac{4}{z^5} \).

Page 76

1(a) When \( z = x + iy \) we have \( f(z) = (3x + y) + i(3y - x) = u(x, y) + iv(x, y) \). Then, observe that \( u_x = 3, u_y = 1 \) and \( v_x = -1, v_y = 3 \). Then, clearly all partial derivatives are continuous and it’s easy to see that \( u_x = v_y \) and \( u_y = -v_x \) for all \( x, y \in \mathbb{R} \). Hence, \( f \) is differentiable everywhere, hence entire and \( f'(z) = u_x + iv_x = 3 - i \).

2(a) Let, \( f(z) = xy + iy = u(x, y) + iv(x, y) \) and observe that \( u_x = y, u_y = x \) and \( v_x = 0, v_y = 1 \). Assume that \( f \) is analytic at some point \( P \) on \( \mathbb{C} \). Then, in some neighbourhood of \( P \), Cauchy -Riemann equations should be satisfied. Hence, in that neighbourhood \( y = 1 \) and \( x = 0 \). But, this is a single point and this is not satisfied by any neighbourhood. Contradiction !! Hence, \( f \) is nowhere analytic.

Page 79

1. If \( f \) is analytic on \( D \) it satisfies the Cauchy-Riemann equations and \( u, v \) twice continuously partially differentiable. So, we have \( ru_r = v_\theta \) and \( \frac{1}{r} u_\theta = -v_r \). Let’s partially differentiate above two equations by \( r \) and \( \theta \) respectively.
Then, we get \( ru_{rr} + u_r = v_{\theta r} \) and \( -\frac{1}{r} u_{\theta \theta} = v_{r \theta} \). But, we know that \( v_{r \theta} = v_{\theta r} \). Hence, \( ru_{rr} + u_r = -\frac{1}{r} u_{\theta \theta} \). Then, we have \( r^2 u_{rr} + ru_r + u_{\theta \theta} = 0 \) as desired.

And also, as \( f \) is analytic on \( D \), so is \( -if = v - iv \). Then, \( v \) is the real part of the analytic function \( -if \). Hence, by the above result \( v \) should also satisfy the same equation.