Problem set

1(b) As \( z(t) = 2e^{it} \) we have \( z'(t) = 2ie^{it} \). So, \( \int_C \frac{z^2 + 2}{z} \, dz = \int_\pi^{2\pi} \frac{2e^{it} + 2(2ie^{it})}{2e^{it}} \, dt = 2i \int_\pi^{2\pi} (e^{it} + 1) \, dt = 2i[\frac{e^{it}}{i} + t]_\pi^{2\pi} = 4 + 2\pi i. \)

4. Suppose \( z \) is in the given semi circle. So, \( z = z(t) = e^{it} \) where \( t \in [0, \pi] \). Then, \( \log(z) = \log(e^{it}) = it \). Therefore, on this path, \( f(z(t)) = e^{-t} \) and also \( z'(t) = ie^{it} \). Hence, \( \int_C f(z) \, dz = \int_0^\pi e^{-t} ie^{it} \, dt = i \int_0^\pi e^{(i-1)t} \, dt = i[\frac{e^{(i-1)t}}{i-1}]_0^\pi = -\frac{1+e^{-\pi}}{2}(1+i). \)

8. Observe that \( \left| \frac{2z^2 - 1}{z^2 + 5z^2 + 4} \right| = \frac{|2z^2 - 1|}{|z^2 + 5z^2 + 4|} \leq \frac{|2z^2 + 1|}{|z^2 + 5z^2 + 4|} \) when \( |z| > 2 \) by triangle inequality. So, when \( |z| = R \) we have \( \left| \frac{2z^2 + 1}{z^2 + 5z^2 + 4} \right| \leq \frac{2R^2 + 1}{R^2 - 1}. \) And, the length of the upper half circle with radius \( R \) is \( \pi R \). Therefore, we have the desired result \( \int_C \frac{z^2 + 2}{z} \, dz \leq \frac{\pi R(2R^2 + 1)}{(R^2 - 1)}. \) So, \( \int_C \frac{z^2 + 2}{z} \, dz \leq \frac{\pi(2^2 + 1/R^2)}{R(1-4/R^2)(1-1/R^2)} \) and clearly RHS tends to 0 as \( R \) tends to infinity. Hence, so is LHS as desired.

10(b) Observe that \( \frac{d}{dz}(2\sin(z/2)) = \cos(z/2) \). Hence, we have \( \int_0^{\pi+2i} \cos(z/2) \, dz = 2\sin((\pi + 2i)/2) - 2\sin(0/2) = 2\cos(i) = e + \frac{1}{e}. \)

10(c) Observe that \( \frac{d}{dz} \left( z - \frac{3}{4} \right)^4 = (z - 2)^3 \). Hence, we have \( \int_1^3 (z - 2)^3 \, dz = \frac{(3-2)^4}{4} - \frac{(1-2)^4}{4} = 0. \)