Q1: Let $S = \{0 < |z-1| < 1\}$. Find the interior, exterior and boundary points of S. Then find the closure of S.

Recall: Let $S \subseteq \mathbb{C}$, $z_0 \in \mathbb{C}$

z_0 must be in S

① z_0 is an interior pt of S if

$\exists$ a neighborhood $\{|z-z_0| < \varepsilon\}$ entirely contained in S

z_0 must be in S^c

② z_0 is an exterior pt of S if

$\exists$ $\{|z-z_0| < \varepsilon\}$ entirely contained in S^c

complement of S

③ z_0 is a boundary pt of S if

z_0 is neither an interior pt nor

an exterior pt of S

Remark: one and precisely one of the above ① ② ③ cases must occur.

☆ {boundary pts} = $\mathbb{C}$ − {interior pts} − {exterior pts}

Closure of S = S ∪ {boundary pts}

In Q1, $S = \{0 < |z-1| < 1\}$

{interior pts of S} $= S = \{0<|z-1|<1\}$

{exterior pts of S} = {interior pts of S^c} $= \{|z-1|>1\}$

∂S = {bdry pts of S} $= \{1\} \cup \{|z-1|=1\}$.

closure of $S = S \cup \partial S = \{|z-1| \le 1\}$

Q2: Find all cubic roots of i and write them in the rectangular form $x+iy$ (Hint: $\sin\frac{\pi}{6} = \sin\frac{5\pi}{6} = \frac{1}{2}$)

A: The Q askes to solve $z^3 = i$

Step 1: Write RHS in polar form $\frac{\pi}{2}+2k\pi$

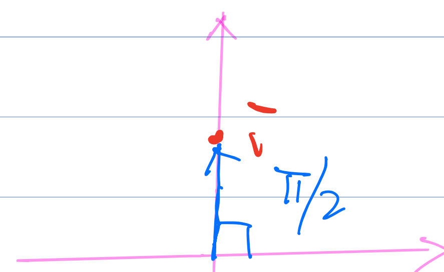
Hint: if $z_0 = x_0 + iy_0$

polar form: $z_0 = r_0 e^{i\theta_0}$

$r_0 = |z_0| = \sqrt{(x_0)^2 + (y_0)^2}$

θ_0: normally taken to be

$\operatorname{Arg} z_0 \ (-\pi < \theta_0 \le \pi)$



$i = 1 \cdot e^{i\pi/2}$

Step 2: write $z = re^{i\theta}$, $\cancel{r>0}$, put it back to the eqn: $z^3 = i$

$$\Rightarrow \quad r^3 e^{3i\theta} = e^{i\frac{\pi}{2}}$$

$$\Rightarrow \begin{cases} r^3 = 1 \\ 3\theta = \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z} \end{cases}$$

$$\Rightarrow \begin{cases} r = 1 \\ \theta = \frac{\pi}{6} + \frac{2}{3}k\pi, \quad k \in \mathbb{Z} \end{cases}$$

$$\Rightarrow \quad z = re^{i\theta} = e^{i\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)}, \quad k = 0, 1, 2$$

Note: In general, when solving $z^n = z_0$, $n \in \mathbb{Z}$, $n \geq 1$.

$k \rightarrow 0, 1, \cdots, n-1$.

$$z = re^{i\theta} = e^{i\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)}, \quad k = 0, 1, 2$$

Step3: finally compute the rectangular form:

$$k=0 \Rightarrow z = e^{i\frac{\pi}{6}} = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{i}{2}$$

$$k=1 \Rightarrow z = e^{i\left(\frac{\pi}{6} + \frac{2}{3}\pi\right)} = e^{i\frac{5}{6}\pi}$$

$$= \cos\left(\frac{5}{6}\pi\right) + i\sin\left(\frac{5}{6}\pi\right)$$

$$= -\frac{\sqrt{3}}{2} + \frac{i}{2}$$

$$k=2 \Rightarrow z = e^{i\left(\frac{\pi}{6} + \frac{4}{3}\pi\right)}$$

$$= e^{i\left(\frac{\pi}{6} + \frac{4}{3}\pi - \overset{\frac{6\pi}{3}}{\cancel{2\pi}}\right)}$$

$$= e^{i\left(\frac{\pi}{6} - \frac{2}{3}\pi\right)}$$

$$= e^{i\left(-\frac{3}{6}\pi\right)} = e^{i\left(-\frac{1}{2}\pi\right)}$$

$$= \overset{0}{\cancel{\cos\left(-\frac{1}{2}\pi\right)}} + i\,\overset{-1}{\cancel{\sin\left(-\frac{1}{2}\pi\right)}}$$

$$= -i$$

Q3. Let S be the horizontal line in $\mathbb{C}$: $S = \{\operatorname{Im} z = 1\}$.

Determine the image of S under the map $w = f(z) = \frac{i}{z}$

A: Step 1: write the expression of $z \in S$ (parametrize S), the compute $f(z)$

$$z \in S \iff z = x + i, \quad x \in \mathbb{R}$$

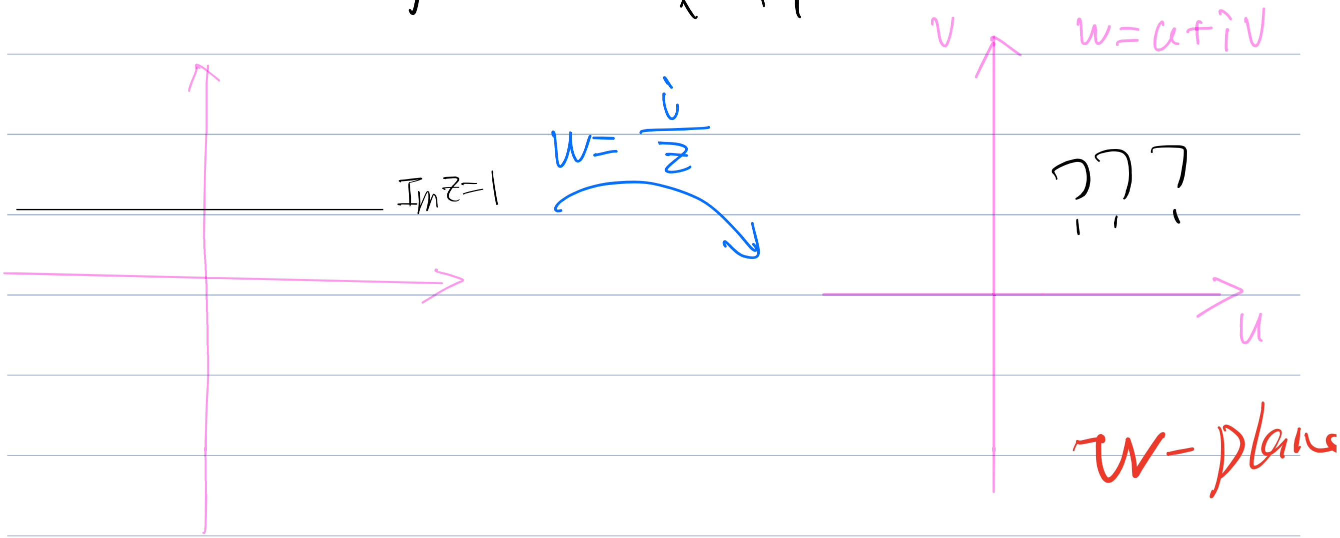
$$\Rightarrow f(z) = \frac{i}{z} = \frac{i}{x+i} = \frac{i(x-i)}{(x+i)(x-i)}$$

$$= \frac{1+ix}{x^2+1}, \quad x \in \mathbb{R}$$

$$= \frac{1}{x^2+1} + i\frac{x}{x^2+1}$$

write $w = f(z) = u(x,y) + iv(x,y)$

$$\Rightarrow \begin{cases} u = \dfrac{1}{x^2+1} & ① \\ \\ v = \dfrac{x}{x^2+1} & ② \end{cases} \qquad x \in \mathbb{R}$$



Step 2: find a relation / formula of u, v

(write x in terms of u, v by ①, ②

then plug into ① or ②)

By ①, ② (divide ② by ①) $\Rightarrow \frac{v}{u} = x$

plug into ① $\Rightarrow$

$$u = \frac{1}{x^2+1} = \frac{1}{\left(\frac{v}{u}\right)^2+1}$$

$$\Rightarrow \quad u = \frac{1}{\left(\frac{v}{u}\right)^2+1} \Rightarrow u = \frac{u^2}{v^2+u^2}$$

divide by u

$$\Rightarrow \quad 1 = \frac{u}{v^2+u^2}$$

or $\quad v^2+u^2 = u$

or $\quad u^2+v^2 = u$

Q: What curve is this ?

$$u^2+v^2 = u$$

$$\Longleftrightarrow (u^2-u) + v^2 = 0$$

$$\Longleftrightarrow \quad \left(u^2 - u + \frac{1}{4}\right) + v^2 = \frac{1}{4}$$

$$\Longleftrightarrow \quad \left(u - \frac{1}{2}\right)^2 + v^2 = \frac{1}{4}$$

circle: center $\left(\frac{1}{2}, 0\right)$

radius $= \sqrt{\frac{1}{4}} = \frac{1}{2}$

Q4: Find the limits and prove by definition (ε-δ language)

(a) $\lim_{z \to 1} (z + \bar{z} - 1)$

Pf: we will show

$$\lim_{z \to 1} \underbrace{(z + \bar{z} - 1)}_{f(z)} = 1 \leftarrow L$$

For any $\varepsilon > 0$, take $\delta = \varepsilon/2$ ''

If $0 < |z-1| < \delta$,

then $|f(z) - L|$

$$= \left| z + \bar{z} - 1 \quad - 1 \right|$$

$$= \left| (z-1) + (\bar{z} - 1) \right|$$

by trig inequality

$$\leq |z-1| + |\bar{z} - 1|$$

$$= |z-1| + |\overline{z-1}|$$

$$= 2|z-1| < 2\delta = 2 \cdot \frac{\varepsilon}{2} = \varepsilon$$

We want: find a good "δ" s.t

when $0 < |z-1| < \delta$

$\Rightarrow |f(z) - L| \leq 2|z-1| < \varepsilon$

Hence by definition,

$$\lim_{z \to 1} f(z) = 1$$

(b) $\lim_{z \to i} (z + \bar{z} - i)$

Read it by yourself

We will show

$$\lim_{z \to i} \underbrace{(z + \bar{z} - i)}_{f(z)} = \underbrace{-i}_{L}$$

Pf: For any $\varepsilon > 0$, take $\delta = \frac{\varepsilon}{2}$.

If $0 < |z - i| < \delta \Longrightarrow$

$$|f(z) - L|$$

$$= |z + \bar{z} - i + i|$$

$$= |(z - i) + (\bar{z} + i)|$$

$$= \left| (z-i) + \overline{(z-i)} \right|$$

$$\leq |z-i| + |\overline{z-i}|$$

$$\leq 2|z-i| < 2\delta = \varepsilon.$$

Hence by definition,

$$\lim_{z \to i} f(z) = -i$$

Q5: Let $f(z) = |z|^2$. ☆☆☆

(a) Show that f is differentiable at 0

(b) Show that f is NOT diff'ble at every $z \neq 0$.

2 ways: use definition.

Cauchy-Riemann eqns.

We will do it by definition

f is $\mathbb{C}$-diff'ble at z $\iff$ $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ exists

We first compute

$f(z) = |z|^2$

$f(z+\Delta z) = |z+\Delta z|^2$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$|w|^2 = w \cdot \overline{w}$

$= u^2 + v^2$

$$= \frac{|z+\Delta z|^2 - |z|^2}{\Delta z}$$

$$= \frac{(z+\Delta z)(\overline{z}+\overline{\Delta z}) - z\overline{z}}{\Delta z}$$

$$= z\frac{\overline{\Delta z}}{\Delta z} + \overline{z} + \overline{\Delta z}$$

(a) At $z=0$,

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} = \overline{\Delta z}$$

$$\Rightarrow \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \overline{\Delta z} = 0$$ exists!

Hence f is $\mathbb{C}$-diff^{ble} at $z=0$,

$$f'(0) = 0.$$

(b) when $z \neq 0$, $\Rightarrow$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(\bar{z} + \overline{\Delta z} + z \frac{\overline{\Delta z}}{\Delta z} \right)$$

Idea: pick 2 directions and compute the limits along these directions and show along these directions. they are NOT equal

Direction 1:

first let $\Delta z \to 0$ along x-axis,

that is, $\Delta z = \Delta x + i0 = \Delta x \in \mathbb{R}$

$\Rightarrow$

the above limit $= \lim_{\Delta x \to 0} \left(\bar{z} + \overline{\Delta x} + z \frac{\overline{\Delta x}}{\Delta x} \right)$

$\Downarrow$

$$= \lim_{\Delta x \to 0} \left(\bar{z} + \Delta x + z \right) \qquad \frac{\Delta x}{\Delta x} = 1$$

$$= \bar{z} + z \qquad ①$$

Direction 2:

Secondly let $\Delta z \to 0$ along y-axis

that is, $\Delta z = 0 + i\Delta y = i\Delta y$

$\Rightarrow$ (Note: $\overline{i\Delta y} = -i\Delta y$)

the above limit $= \lim\limits_{\Delta y \to 0} \left(\overline{z} + \overline{i\Delta y} + z \dfrac{\overline{i\Delta y}}{i\Delta y} \right)$

$= \lim\limits_{\Delta y \to 0} \left(\overline{z} - i\Delta y + z \underbrace{\dfrac{-i\Delta y}{i\Delta y}}_{-1} \right)$

$= \overline{z} - z$ ②

Q: Does ① equal to ② ?

(notice ① = ②

$\Leftrightarrow \overline{z} + z = \overline{z} - z$

$\Leftrightarrow 2z = 0 \Leftrightarrow z = 0$

But we assumed $z \neq 0$)

A: No.

Hence $f'(z)$ DNE at $z \neq 0$.