Recall last lecture on Cauchy-Riemann eqns

Thm: Let $f : S \subseteq \mathbb{C} \to \mathbb{C}$

Let $z_0 = x_0 + iy_0 \in S$

Write $f(z) = u(x,y) + iv(x,y)$

1 u, v are $\mathbb{R}$-diff^ble at (x_0, y_0)

$+$

2 $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ at (x_0, y_0)

C.-R. eqns

$\Longleftrightarrow$ $f(z)$ is $\mathbb{C}$-diff^ble at $z_0 = x_0 + iy_0$

In this case, $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$

$$= v_y(x_0, y_0) - iu_y(x_0, y_0)$$

Last time we used to the above

Thm to show

E.g let $f(z) = z^2$. $\Rightarrow f'(z) = 2z$

Today, we will do more E.gs.

E.g

① $f(z) = \bar{z} = x - iy = x + i(-y)$

Does $f'(z)$ exist?

A:

Step 1: find u, v and

compute u_x, u_y, v_x, v_y.

Note: $\begin{cases} u = x \\ v = -y \end{cases}$

Notation:
$u_x = \frac{\partial u}{\partial x}$
$u_y = \frac{\partial u}{\partial y}$

$\Longrightarrow \begin{cases} u_x = 1 \\ u_y = 0 \end{cases}$; $\begin{cases} v_x = 0 \\ v_y = -1 \end{cases}$

Step 2: check whether the C.-R. eqns hold.

Check

$\begin{cases} u_x \stackrel{?}{=} v_y \quad (\text{i.e. } 1 \stackrel{?}{=} -1) \times \\ u_y \stackrel{?}{=} -v_x \end{cases}$

Thus. the C.-R. eqns do NOT hold. $\Longrightarrow$ $f(z) = \bar{z}$ is NOT $\mathbb{C}$-diff^{ble} at every $z \in \mathbb{C}$.

(2) $f(z) = |z|^2 = \underbrace{x^2 + y^2}_{u} + i \cdot \underbrace{0}_{v}$

A: Step 1

Note: $\begin{cases} u = x^2 + y^2 \\ v = 0 \end{cases}$

$\Longrightarrow \begin{cases} u_x = 2x \\ u_y = 2y \end{cases}$; $\begin{cases} v_x = 0 \\ v_y = 0 \end{cases}$

Step 2:

Check C.-R. eqns:

$x=0$

$\Updownarrow$

$$\begin{cases} u_x = v_y \text{ (i.e. } 2x=0 \text{?)} \\ u_y \overset{?}{=} -v_x \text{ (i.e. } 2y=0 \text{?)} \end{cases}$$

$\Updownarrow$

$y=0$

Note: The C.-R. eqns only hold at $(x,y)=(0,0)$

$\Longrightarrow f(z)=|z|^2$ is only $\mathbb{C}$-diff^{ble} at $z_0 = 0+i0 = 0$. Moreover,

$f'(0) = u_x + i v_x = 0 + i \cdot 0 = 0$

③ $f(z) = e^z$ (here $z = x+iy$)

$\triangleq e^x e^{iy}$

$= e^x(\cos y + i \sin y)$

$= e^x \cos y + i e^x \sin y$

A: Step 1:

$$\begin{cases} u = e^x \cos y \\ v = e^x \sin y \end{cases}$$

$\frac{d}{dx} e^x = e^x$

$\frac{d}{dy} \cos y = -\sin y$

$$\Rightarrow \begin{cases} u_x = e^x \cos y \\ u_y = -e^x \sin y \end{cases} \quad \begin{cases} v_x = e^x \sin y \\ v_y = e^x \cos y \end{cases}$$

Step 2:

Check C.-R. eqns

$$\begin{cases} u_x = v_y \ (\text{i.e. } e^x \cos y = e^x \cos y \, ?) \\ u_y = -v_x \ (\text{i.e. } -e^x \sin y = -e^x \sin y) \end{cases}$$

$\Rightarrow$ C.-R. eqns hold at every pt in $\mathbb{C}$

Hence, $f'(z)$ exists at every pt in $\mathbb{C}$.

Moreover, with $f(z) = e^z$

$$f'(z) = u_x + i v_x$$

$$= e^x \cos y + i e^x \sin y$$

$$= e^x (\cos y + i \sin y)$$

$$= e^x e^{iy} = e^z$$

(here $z = x + iy$)

complex exponential

That is, $\frac{d}{dz} e^z = e^z$

Remark: This is very similar to Calculus:

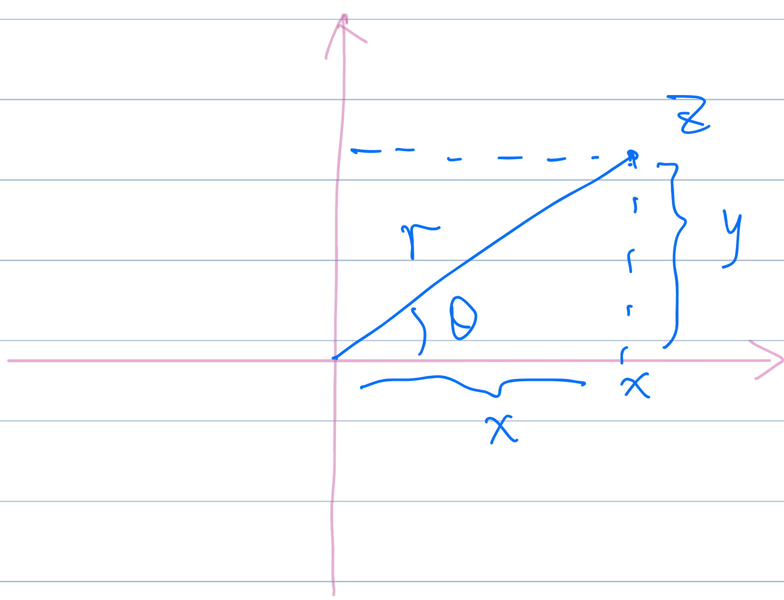
$$\frac{d}{dx} e^x = e^x$$

real exponential

Q: The above C.-R. eqns are in the $\underbrace{\text{rectangular form}}_{z=x+iy}$. Can we write C.-R. eqns in polar coordinates?

A: Yes!

To do that, recall Calculus:



$$z = x + iy$$
$$= re^{i\theta}$$

$$\Rightarrow \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

write $f(z) = u(\overset{r\cos\theta}{x}, \overset{r\sin\theta}{y}) + i\,v(\overset{r\cos\theta}{x}, \overset{r\sin\theta}{y})$

$= u(r,\theta) + i\,v(r,\theta)$

write $u_x = \dfrac{\partial u}{\partial x}$, $u_y = \dfrac{\partial u}{\partial y}$

$u_r = \dfrac{\partial u}{\partial r}$, $u_\theta = \dfrac{\partial u}{\partial \theta}$

set 1: $\begin{cases} u_x \\ u_y \end{cases}$

set 2: $\begin{cases} u_r \\ u_\theta \end{cases}$

Q: Can we express u_r, u_θ in terms of u_x, u_y?

(Hint: $u(x, y) = u(r\cos\theta, r\sin\theta)$)

A: Yes! Use Chain Rule from Calculus

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Indeed,

$$u_r = \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

$$= u_x \cos\theta + u_y \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

∂v

Similarly,

chain Rule

$$u_\theta = \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= -u_x r\sin\theta + u_y r\cos\theta$$

Likewise,

$$v_r = v_x \cos\theta + v_y \sin\theta$$

$$v_\theta = -v_x r\sin\theta + v_y r\cos\theta$$

Fact: When $z \neq 0$, i.e. $r \neq 0$, we have

$$\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}} \Longleftrightarrow \boxed{\begin{aligned} u_r &= \frac{1}{r} v_\theta \quad (1) \\ v_r &= -\frac{1}{r} u_\theta \quad (2) \end{aligned}}$$

C.-R. eqns in rectangular form

C.-R. eqns in polar form

$$\text{LHS of (1)} = u_r$$

Pf of (1):

$$= u_x \cos\theta + u_y \sin\theta$$

$$= v_y \cos\theta - v_x \sin\theta$$

$$\text{RHS of (1)} = \frac{1}{r} v_\theta$$

$$= \frac{1}{r}\left(-v_x r\sin\theta + v_y r\cos\theta\right)$$

$$= -v_x \sin\theta + v_y \cos\theta$$

Hence (1) is proved!

Pf of (2): Ex.

Remark: Most times we use C.-R. eqns in x-y coordinates, but sometimes C.-R. eqns in polar coordinates can be easier.

Thm (of C.-R. eqns). Write $f(z) = u(r,\theta) + i v(r,\theta)$, $z \neq 0$.

Then at $z_0 = r_0 e^{i\theta_0}$

① $\boxed{u, v \text{ are } \mathbb{R}\text{-diff}^{ble} \text{ at } (r_0, \theta_0)}$

+

② $\boxed{\begin{array}{l} u_r = \frac{1}{r} v_\theta \\ v_r = -\frac{1}{r} u_\theta \end{array} \text{ at } (r_0, \theta_0)}$

$\Longleftrightarrow$ $\boxed{f(z) \text{ is } \mathbb{C}\text{-diff}^{ble} \text{ at } z_0 = r_0 e^{i\theta_0}}$

C.-R. eqns in polar form

In this case, $f'(z_0) = e^{-i\theta}(u_r + i v_r)\Big|_{z_0 = r_0 e^{i\theta_0}}$ (*)

Pf of (*) : (NOT required)

Recall: $f'(z) = u_x + i v_x$

on the other hand,

RHS of (*) $= \underbrace{e^{-i\theta}}_{= \cos(-\theta) + i\sin(-\theta)}(u_r + i v_r)$

$= (\cos\theta - i\sin\theta)\big(u_x\cos\theta + u_y\sin\theta$

$+ i(v_x\cos\theta + v_y\sin\theta)\big)$

$= (u_x\cos^2\theta + u_y\sin\theta\cos\theta + \cancel{v_x}\sin\theta\cos\theta$ $-u_y$

$+ \cancel{v_y}\sin^2\theta)$ u_x

$\boxed{\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}}$

$$+\, i\left(\overset{-U_y}{\cancel{V_x}}\cos^2\theta + \overset{U_x}{\cancel{V_y}}\sin\theta\cos\theta - U_x\sin\theta\cos\theta\right.$$

$$\left. - U_y\sin^2\theta\right)$$

$$= U_x\left(\overset{1}{\cancel{\cos^2\theta + \sin^2\theta}}\right) + i(-U_y)\left(\overset{1}{\cancel{\sin^2\theta + \cos^2\theta}}\right)$$

$$= U_x - iU_y = U_x + iV_x$$