Complex derivative

Def^n: Let $f: S \subseteq \mathbb{C} \to \mathbb{C}$

z_0: an interior pt of S.

f is (complex) differentiable at z_0

$\Longleftrightarrow$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \text{ exists}$$

exists as a complex number (NOT ∞)

Two ways to check if f' exists

(A) Check by Definition

(B) Check by Cauchy-Riemann equations

Two ways to express Cauchy-Riemann equations

(I) Use rectangular coordinates

Thm: Let $f: S \subseteq \mathbb{C} \to \mathbb{C}$

Let $z_0 = x_0 + iy_0 \in S$

Write $f(z) = u(x,y) + iv(x,y)$

① u, v are $\mathbb{R}$-diff^ble at (x_0, y_0)

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② $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ at (x_0, y_0)

C.-R. eqns

$\iff$ $f(z)$ is $\mathbb{C}$-diff^ble at $z_0 = x_0 + iy_0$

In this case, $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$

$$= v_y(x_0, y_0) - iu_y(x_0, y_0)$$

(II) Use polar coordinates

Thm. Write $f(z) = u(r,\theta) + i v(r,\theta)$, $z \neq 0$.

Then at $z_0 = r_0 e^{i\theta_0}$

① $\boxed{u, v \text{ are } \mathbb{R}\text{-diff}^{ble} \text{ at } (r_0, \theta_0)}$ $u_r, u_\theta, v_r, v_\theta$

$+$

② $\boxed{\begin{array}{l} u_r = \frac{1}{r} v_\theta \\ v_r = -\frac{1}{r} u_\theta \end{array} \text{ at } (r_0, \theta_0)}$

$\iff$ $\boxed{f(z) \text{ is } \mathbb{C}\text{-diff}^{ble} \text{ at } z_0 = r_0 e^{i\theta_0}}$

C.-R. eqns in polar form

In this case, $f'(z_0) = e^{-i\theta}(u_r + i v_r)\Big|_{z_0 = r_0 e^{i\theta_0}}$

Remark: Most times we use C.-R. eqns in x-y coordinates, but sometimes C.-R. eqns in polar coordinates can be easier.

Here is an example:

E.g 1. Let $f(z) = \frac{1}{z}$

prove $f'(z) = -\frac{1}{z^2}$

without using quotient rule.

Hint: use C.-R. eqns in polar form

Pf: Step 1. Find u, v in polar coordinates, then compute $u_r, u_\theta, v_r, v_\theta$.

$f(z) = \frac{1}{z}$ write $z = re^{i\theta}$

$$\Rightarrow f(z) = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}$$

Hint: $e^{-i\theta} = e^{i(-\theta)}$

$= \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$

$$\Rightarrow f(z) = \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$= \frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta$$

Hence $\begin{cases} u = \frac{1}{r}\cos\theta \\ v = -\frac{1}{r}\sin\theta \end{cases}$

$$\Rightarrow \begin{cases} u_r = \frac{\partial u}{\partial r} = -\frac{1}{r^2}\cos\theta \\ u_\theta = \frac{\partial u}{\partial \theta} = \frac{1}{r}(-\sin\theta) \end{cases}$$

$$\begin{cases} V_r = \frac{1}{r^2}\sin\theta \\ V_\theta = -\frac{1}{r}\cos\theta \end{cases}$$

Step 2: Check C.-R. eqns hold or not

$$\begin{cases} U_r = \frac{1}{r}V_\theta \quad \checkmark \\ V_r = -\frac{1}{r}U_\theta \quad \checkmark \end{cases}$$

Hence C.-R. eqns holds.

Step 3. Compute $f'(z)$

$$f' = e^{-i\theta}(u_r + i v_r)$$

$$= e^{-i\theta}\left(-\frac{\cos\theta}{r^2} + i\,\frac{\sin\theta}{r^2}\right)$$

$$= \frac{e^{-i\theta}}{r^2}(-\cos\theta + i\sin\theta)$$

$$= -\frac{e^{-i\theta}}{r^2}\underbrace{(\cos\theta - i\sin\theta)}_{e^{-i\theta}}$$

$$= -\frac{e^{-i\theta}}{r^2}\, e^{-i\theta}$$

$$= -\frac{1}{r^2 e^{2i\theta}} = -\frac{1}{(re^{i\theta})^2}$$

$$\boxed{\begin{array}{l}\text{Recall} \\ z = re^{i\theta}\end{array}}$$

$$= -\frac{1}{z^2}$$

Hence

if $f = \frac{1}{z} \Rightarrow f' = -\frac{1}{z^2}$

Eg 2: Let $f(z) = \frac{1}{z^2}$

Compute $f'(z)$ $\left(= -2 \frac{1}{z^3} \right)$

by using C.-R. eqns in polar coordinates

Ex!

Analytic functions (P72 in book)

Def^n: Let $S \subseteq \mathbb{C}$ be open

($\Leftrightarrow$ every pt $z \in S$ is an interior pt of S)

We say f is analytic in S

$\Leftrightarrow$ $f'(z)$ exists for all $z \in S$

complex derivative

Note: Some textbooks use the term "holomorphic" instead of "analytic" to mean the same thing.

Def^n: let $S \subseteq \mathbb{C}$ (not necessarily open)

$z_0 \in S$ an interior pt:

$f: S \to \mathbb{C}$

we say f is analytic at z_0

$\iff$ f is analytic on some $\underbrace{\text{neighborhood}}_{\{|z-z_0|<\varepsilon\}}$ of z_0

$\iff$ $f'(z)$ exists at all points in $\{|z-z_0|<\varepsilon\}$

Note:

① Sums and products of analytic functions are analytic

② Quotient of two analytic functions

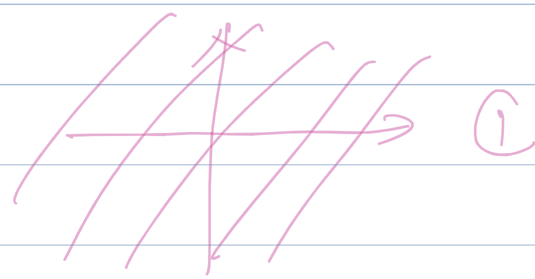
$\frac{f}{g}$ are analytic, given $g(z_0) \neq 0$

③ Compositions of analytic functions are analytic.

Def^n: $f(z)$ is an entire function

$\Longleftrightarrow$ $f(z)$ is analytic at every pt $z \in \mathbb{C}$

($\Leftrightarrow$ $f'(z)$ exists at every pt $z \in \mathbb{C}$)



Remark:

Compositions of entire functions are entire.

E.g: ① polynomials in z:

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

(all $a_j \in \mathbb{C}$.) is entire

② Complex exponential

$f(z) = e^z$ is entire

(Recall: $\frac{d}{dz} e^z = e^z$)

③ Let p: a polynomial in z

$g(z) = e^{p(z)}$ is entire

E.g e^{z^2+z+2} is entire.

$\Rightarrow \frac{f(z)}{g(z)}$

E.g: Let $h(z) = \frac{1}{z^2+1}$. Is it entire?

Note: $h = \frac{1}{z^2+1} = \frac{1}{(z-i)(z+i)}$

is only analytic $\mathbb{C} - \{i, -i\}$

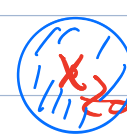
Hence $h(z)$ is NOT entire.

Def^n: If $f(z)$ is analytic on

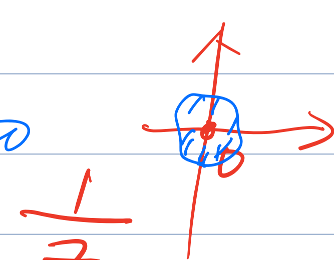
$\{0 < |z - z_0| < \varepsilon\}$ (deleted nbhd of z_0),

but not analytic at z_0.

then z_0 is called a singularity of f.



E.g: ① $f(z) = \frac{1}{z}$ has a singularity at $z = 0$



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Recall:

fact from Calculus:

$f(x) : I = (a,b) \to \mathbb{R}$

Assume $f'(x) = 0$ everywhere in I

$\Longleftrightarrow$ f is constant.

$\Leftrightarrow$ f takes the same value at every pt in I.

Q: Do we have an analogous fact in complex analysis?

Thm: Let $f(z)$ be an analytic function on a $\underbrace{\text{domain}}_{\text{"open + connected"}}$ $D \subseteq \mathbb{C}$

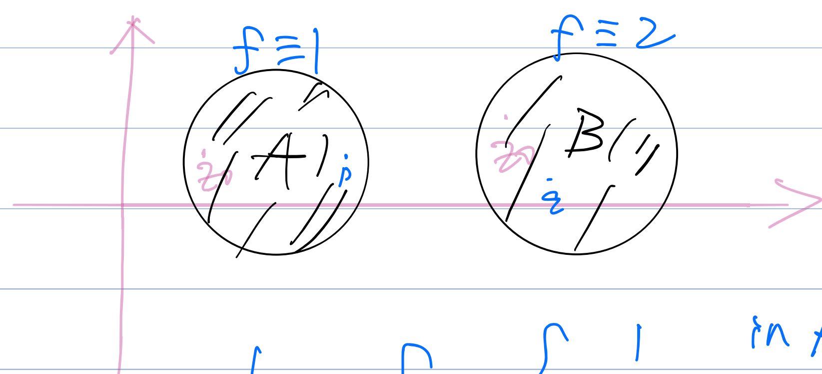
If $f'(z) = 0$ at all $z \in D$

$\Rightarrow$ f is $\underline{\text{constant}}$ in D

$\Leftrightarrow$ f takes the same value at every pt in D.

Remark: If D is not connected, then the conclusion fails.

E.g.: Let $D = A \cup B$ $\leftarrow$ D is open but not connected



Let $f = \begin{cases} 1 & \text{in } A \end{cases}$

let $J -]_2$ in B.

$\Rightarrow f$ is constant on A, but NOT constant
is constant on B; on D

and $f' \equiv 0$ in A and $B \Rightarrow f' \equiv 0$
on D

Pf of Thm:

Recall Calculus of several variables (SoC)

let $D \subseteq \mathbb{R}^2$ (open + connected) domain

If $u(x,y): D \to \mathbb{R}$ a $\mathbb{R}$-diff'ble function,

Assume the gradient

$$\nabla u = (u_x, u_y) \equiv 0 \text{ everywhere}$$

in $D \Longrightarrow u$ is constant in D

Pf of Thm:

By assumption. $f'(z) = 0$
at all $z \in D$.

Recall C-R. eqns:

writing $f = u + iv$, since $f'(z) = 0$

$\Rightarrow$ $\boxed{u, v \text{ are } \mathbb{R}\text{-diff'ble in } D}$ + $\boxed{\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}}$

and $f'(z) = u_x + i v_x$

$\Rightarrow$ $f'(z) = u_x + i v_x = 0$

$$\Rightarrow \begin{cases} u_x = 0 \overset{\text{C-R. eqns}}{\Longrightarrow} v_y = 0 \\ v_x = 0 \Rightarrow u_y = 0 \end{cases}$$

$\nabla u = (u_x, u_y) = 0 \Rightarrow u$ is constant in D

$\nabla v = (v_x, v_y) = 0$

$\Rightarrow v$ is constant in D

$f = u + iv$ is constant