Harmonic functions

Defn Let $\Omega \subseteq \mathbb{R}^2$ open $\Rightarrow$ u_x, u_y, u_{xx}, u_{xy}, u_{yy} exist

Let $\underset{}{u} = u(x,y)$: $u: \Omega \to \mathbb{R}$ twice $\mathbb{R}$-diff^{ble}

We say u is harmonic in Ω

$\Longleftrightarrow$ $\underline{u_{xx} + u_{yy} = 0}$ everywhere in Ω.

☆ "Laplace's eqn"

Remark:

$$u_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial x^2}$$

$$u_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial y^2}$$

Thm: If $f(z) = u(x,y) + iV(x,y)$ is analytic in a domain $D \subseteq \mathbb{C}$, then both $u(x,y)$ and $V(x,y)$ are harmonic in D

Pf. (NOT required)

Recall:

V_u exBes

Thm

$$\left.\begin{array}{l} ① \ \boxed{u, v \text{ are } \mathbb{R}\text{-diff'ble}} \iff u_x, u_y, v_x, v_y \\[2ex] ② \ \boxed{\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}} \end{array}\right\} \iff \boxed{f \text{ is } \mathbb{C}\text{-diff'ble}}$$

Remark:

We will show in a later lecture that f is analytic in D $\Rightarrow$ u, v have partial derivatives of any order. For now, we just assume that second order partial derivatives exist.

Recall: C.-R. eqns:

$$\begin{cases} u_x = v_y & ① \\ u_y = -v_x & ② \end{cases}$$

Take $\frac{\partial}{\partial x}$ of ① $\Rightarrow$

$$u_{xx} = \frac{\partial}{\partial x}(v_y)$$

$$= v_{yx} = v_{xy} \quad ③$$

Take $\frac{\partial}{\partial y}$ of ② $\Rightarrow$

$$u_{yy} = -v_{xy} \quad ④$$

Add up ③ + ④ $\Rightarrow$

$$u_{xx} + u_{yy} = 0$$

Hence, u is harmoniz in D.

Similarly, take $\frac{\partial}{\partial y}$ of ① $\Rightarrow u_{xy} = v_{yy}$

take $\frac{\partial}{\partial x}$ of ② $\Rightarrow u_{xy} = -v_{xx}$

Substract them $\Rightarrow$ $0 = v_{xx} + v_{yy}$

E.g.: Let $v(x,y) = e^x \sin y$. Prove $v(x,y)$ is harmonic on $\mathbb{R}^2$.

Pf.: There are 2 ways to prove.

Way 1: Use definition (verify $\begin{cases} v \text{ twice } \mathbb{R}\text{-diffble} \\ v_{xx} + v_{yy} = 0 \end{cases}$)

We need to compute v_{xx}, v_{yy}.

Note: $v(x,y) = e^x \sin y$

$\Rightarrow \quad v_x = e^x \sin y, \quad v_y = e^x \cos y$

$\Rightarrow \quad v_{xx} = \frac{\partial}{\partial x}(v_x) = e^x \sin y$

$v_{yy} = \frac{\partial}{\partial y}(v_y) = e^x(-\sin y)$

$\Rightarrow \quad v_{xx} + v_{yy} = 0 \Rightarrow v$ is harmonic in $\mathbb{R}^2$

Way 2: Use the above Thm

Consider $f(z) = e^z$

complex exponential:

$f'(z)$ exists, $f'(z) = e^z$

Recall: $f(z)$ is analytic in $\mathbb{C}$.

Note: $f(z) = e^z = e^{x+iy}$

$$\overset{\triangle}{=} e^x e^{iy}$$

$$= e^x (\cos y + i \sin y)$$

$$= \underbrace{e^x \cos y}_{u} + i \underbrace{e^x \sin y}_{v}$$

$\overset{\text{by Thm}}{\Longrightarrow}$ u, v are harmonic in $\mathbb{C} \cong \mathbb{R}^2$

Def^n: Let u, v be harmonic in D, we say v is a harmonic conjugate of u $\iff$ $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ in D

(i.e., C.-R. eqns hold for u, v)

Thm: Let u, v be harmonic in D. Then v is a harmonic conjugate of u $\iff$ $f = u + iv$ is analytic in D

E.g.: $\underbrace{e^x \sin y}_{v}$ is a harmonic conjugate of $\underbrace{e^x \cos y}_{u}$ on $\mathbb{C}$

because: $f(z) = e^x \cos y + i e^x \sin y$
$= e^{x+iy} = e^z$
is analytic on $\mathbb{C}$

Remarks:

- If f is analytic, we say its imaginary part is a harmonic conjugate of the real part

- If v is a harmonic conjugate of u $\Rightarrow$

  $-u$ is a harmonic conjugate of v

(Warning: In this case, you cannot say u is a harmonic conjugate of v)

Why? By assumption

Why: By assumption

$f = u + iv$ is analytic

$\Rightarrow g \triangleq -if$ is analytic $= -i \cdot f$

$\Rightarrow g = -i(u+iv) = -iu + v$

$= v + i(-u)$ is analytic

Hence $-u$ is a harmonic conjugate of v.

Q: Let $u(x,y) = 2xy$ on $\mathbb{R}^2$.

① Verify u is harmonic on $\mathbb{R}^2$

② Find all harmonic conjugates of u on $\mathbb{R}^2$.

A: ① (verify $u_{xx} + u_{yy} = 0$)

Note: $u = 2xy$

$$\Rightarrow \begin{cases} u_x = 2y \\ u_y = 2x \end{cases} \Rightarrow \begin{cases} u_{xx} = 0 \\ u_{yy} = 0 \end{cases}$$

$$\Rightarrow u_{xx} + u_{yy} = 0$$

② (Find all harmonic conjugates of u on $\mathbb{R}^2$)

Idea: find all v on $\mathbb{R}^2$ such that

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

Step 1: Use C.-R. eqns to find v_x, v_y.

Since $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \Rightarrow \begin{cases} v_y = u_x = 2y \\ v_x = -u_y = -2x \end{cases}$

Q: Can you recover $v(x,y)$ from v_x, v_y

A: Yes. E.g. $F'(x) = 2x$

$$\Rightarrow F(x) = \int 2x\,dx = x^2 + C$$

Step 2: Integrate $v_x = -2x$ (by regarding y as a constant)

$$\Rightarrow v = \int v_x\,dx$$

"constant"

(*) $$= \int -2x\,dx = -x^2 + \varphi(y)$$

Step 3: Substitute the formula (*) to $V_y = 2y$

$$\Rightarrow \frac{\partial}{\partial y}(V) = \frac{\partial}{\partial y}\left(-x^2 + \varphi(y)\right) = 2y$$

$$\Rightarrow \quad \varphi'(y) = 2y$$

$$\Rightarrow \quad \varphi = \int \varphi' \, dy = \int 2y \, dy$$

$$= y^2 + C$$

Summary:

$$V = -x^2 + \varphi(y) = -x^2 + y^2 + C$$

is a harmonic conjugate of $u = 2xy$

Remark: We can verify our final answer in the following way:

$$f(z) = u + iV$$

$$= 2xy + i(-x^2 + y^2 + C)$$

$(-i)i = 1$

$$= 2xy + i(-x^2 + y^2) + iC$$

$$= -i\left(i2xy + 1(x^2 - y^2)\right) + iC$$

$$= -i(x + iy)^2 + iC$$

$$= -iz^2 + iC$$

$z = x + iy$

all polynomials are analytic