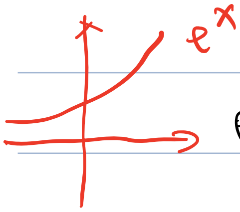
9. Elementary functions

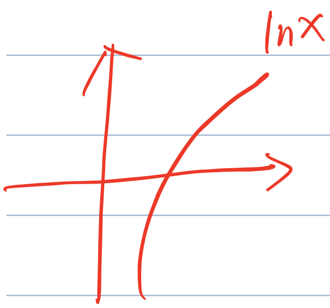
Recall in Calculus / real analysis

- real exponential function: $y = e^x, \ x \in \mathbb{R}$



$$e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}, \quad \frac{d}{dx}(e^x) = e^x$$

- real log function: $y = \ln x = \log_e x, \ x > 0$



Recall: $y = e^x > 0$

$$\Longleftrightarrow \ x = \ln y$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

In complex analysis, do we have analogous functions?

① (complex) exponential function. $w = e^z$

If $z = x + iy$, then

$$e^z \triangleq e^x \cdot e^{iy} \quad (*)$$

$$= e^x(\cos y + i\sin y)$$

Properties of e^z:

① e^z is entire (analytic on $\mathbb{C}$), $\frac{d}{dz} e^z = e^z$

② $|e^z| = |e^x \cdot e^{iy}| = |e^x||e^{iy}|$

$= |e^x| = e^x > 0$

☆ ③ $e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}$

E.x: Use $(*)$

④ $\frac{1}{e^z} = e^{-z}$

why? In ③, put $z_1 = z$, $z_2 = -z$

$$\Rightarrow (e^{z}) \cdot (e^{-z}) = e^{z-z} = e^{0} = 1$$

⑤ $\arg(e^{z}) = y + 2n\pi, \ n \in \mathbb{Z}$

why? $e^{z} = \underbrace{e^{x}}_{r} \cdot \underbrace{e^{iy}}_{e^{i\theta}}$

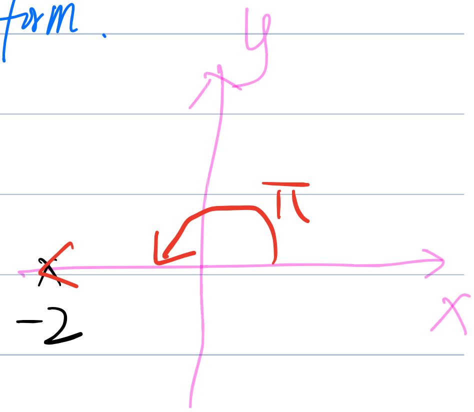
$$e^{z} = e^{x} \cdot e^{iy}$$

$\overbrace{}^{}\ r \qquad \overbrace{}\ e^{i\theta}$

E.g: find all z s.t $e^z = -2$.

Step 1: write -2 in polar form.

$$-2 = re^{i\theta} = 2e^{i\pi}$$



Step 2: Write $z = x + iy$ and plug into $e^z = -2$.

$$e^z = \underbrace{e^x}_{r} \cdot \underbrace{e^{iy}}_{e^{i\theta}} = 2e^{i\pi}$$

$$\Rightarrow \begin{cases} e^x = 2 \Rightarrow x = \ln 2 \\ y = \pi + 2n\pi,\ n \in \mathbb{Z} \end{cases} \Rightarrow \begin{aligned} z &= x + iy \\ &= \ln 2 + i(\pi + 2n\pi) \\ &\quad n \in \mathbb{Z} \end{aligned}$$

Next Q: Can we define the complex analog of log?

A: Yes! but a bit tricky.

Idea: Consider the inverse of $f(z) = e^z$.

Q: Given $z \in \mathbb{C}$, $z \neq 0$, find all $w \in \mathbb{C}$, s.t $e^w = z$.

A: write $z = re^{i\theta}$, $r > 0$, $\theta = \operatorname{Arg} z \in (-\pi, \pi]$

$w = u + iv$

Then $e^w = z \Rightarrow$

$e^u e^{iv} = re^{i\theta}$

$r = |z|$

$$\Rightarrow \begin{cases} e^u = r \Rightarrow u = \ln r = \ln|z| \\ v = \theta + 2n\pi,\ n \in \mathbb{Z} \end{cases}$$

$= \arg z = \operatorname{Arg} z + 2n\pi,\ n \in \mathbb{Z}$

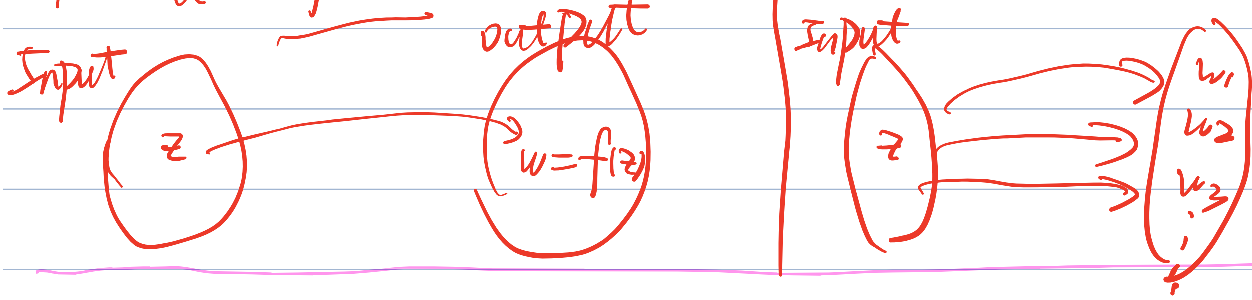
Hence

$e^w = z \Leftrightarrow w = u + iv = \ln|z| + i\arg z$

Def^n: The complex logrithm is the "multi-valued function" defined by

$$\log z \triangleq \ln|z| + i\arg z, \quad z \neq 0$$
$$= \ln|z| + i\operatorname{Arg} z + i2n\pi, \; n \in \mathbb{Z}$$

Recall: function | multi-valued function



Warning: "multivalued function" is NOT a function in the usual sense.

Def^n: The principal value of the complex logarithm is the function is

$$\mathrm{Log}\, z \triangleq \ln|z| + i\underbrace{\mathrm{Arg}\, z}_{\in(-\pi,\pi]}$$

E.g: Find $\log(1+\sqrt{3}i)$ and $\mathrm{Log}(1+\sqrt{3}i)$

Hint: $1+\sqrt{3}i = 2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$
$= 2e^{i\pi/3}$

$$\boxed{\begin{array}{l} \log z = \ln|z| + i\arg z; \\ \mathrm{Log}\, z = \ln|z| + i\mathrm{Arg}\, z; \end{array} \quad z\neq 0}$$

$\log(1+\sqrt{3}i) = \ln|z| + i\arg z$

$= \ln 2 + i\left(\frac{\pi}{3} + 2n\pi\right),\ n\in\mathbb{Z}$

$\mathrm{Log}(1+\sqrt{3}) = \ln|z| + i\mathrm{Arg}\, z \quad \in(-\pi,\pi]$

$= \ln 2 + i\,\pi/3$

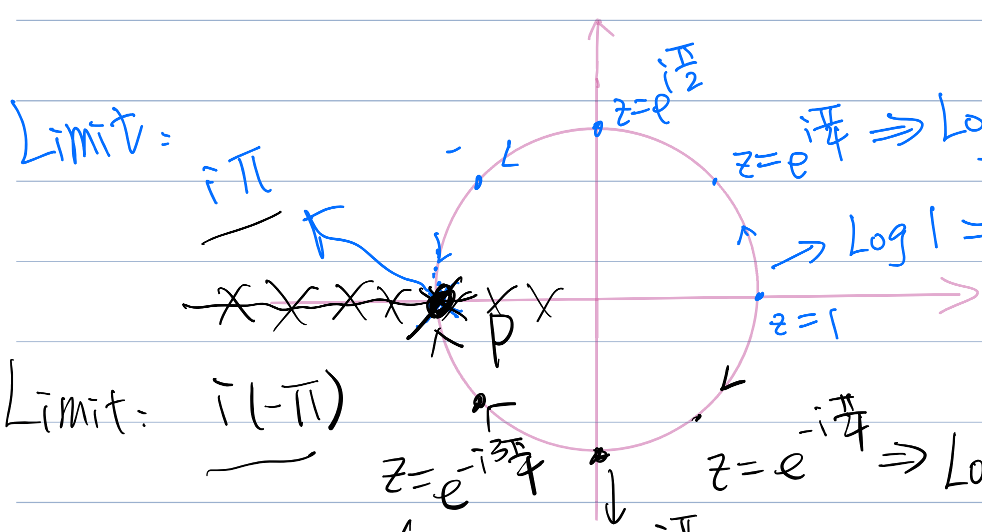
$e^z \longleftrightarrow \mathrm{Log}\, z$

Q: Recall e^z is entire (analytic in $\mathbb{C}$)

Is $\operatorname{Log} z$ also entire?

A: $\operatorname{Log} z$ is NOT analytic on $\mathbb{C}$!

Indeed, $\operatorname{Log} z$ is NOT even continuous on $\mathbb{C}$!



$z=e^{i\frac{\pi}{4}} \Rightarrow \operatorname{Log} e^{i\frac{\pi}{4}} = \ln 1 + i \operatorname{Arg} e^{i\frac{\pi}{4}} = 0 + i\frac{\pi}{4}$

$\Rightarrow \operatorname{Log} 1 = \ln|1| + i\operatorname{Arg} 1 = 0 + i0 = 0$

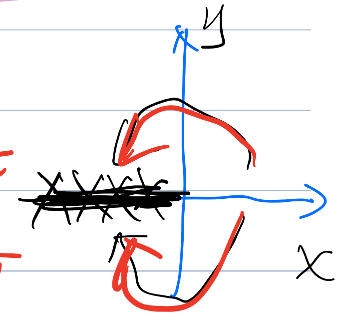
$z=e^{-i\frac{\pi}{4}} \Rightarrow \operatorname{Log} e^{-i\frac{\pi}{4}} = i(-\frac{\pi}{4})$

$\operatorname{Log} e^{-i\frac{3\pi}{4}} = i(-\frac{3\pi}{4})$

$= i(-\frac{\pi}{2})$

Note: • The problem really occurs at

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$$
$$-\pi < \operatorname{Arg} z \leq \pi$$



$(-\pi, \quad \pi]$

$\longleftarrow \quad \longrightarrow$

- To overcome this problem, we delete 0 and delete the negative x-axis from $\mathbb{C}$ $\Rightarrow$

$$D = \{z = re^{i\theta} \mid r > 0, \ -\pi < \theta < \pi\}$$

Then

- $\operatorname{Log} z$ is continuous on D.

- $\operatorname{Log} z$ is analytic on D.

why?

(r, θ)

Note:

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$$
$$= \underbrace{\ln r}_{u} + i \underbrace{\theta}_{v}, \quad \theta \in (-\pi, \pi)$$

$$\Rightarrow \begin{cases} u = \ln r \\ v = \theta \end{cases}$$

(Hint: we will check the complex differentiability using C.-R. eqns in polar form)

① u, v are $\mathbb{R}$-diff'ble on D

② C.-R. eqns in polar form:

Recall
$$\begin{cases} u_r \overset{?}{=} \dfrac{v_\theta}{r} \checkmark \\ v_r \overset{?}{=} -\dfrac{u_\theta}{r} \checkmark \end{cases}$$

Check:
$$\begin{cases} u_r = \dfrac{1}{r} \\ u_\theta = 0 \end{cases} ; \quad \begin{cases} v_r = 0 \\ v_\theta = 1 \end{cases}$$

Note both C.-R. eqns $\Longrightarrow$

$f(z) = \operatorname{Log} z$ is analytic on D

Moreover,

$$f'(z) = \frac{d}{dz}(\operatorname{Log} z)$$

$$= e^{-i\theta}(u_r + i v_r)$$

$$\boxed{\frac{1}{e^{i\theta}} = e^{-i\theta}}$$

$$= e^{-i\theta}\left(\frac{1}{r} + i \cdot 0\right)$$

$$= e^{-i\theta}\frac{1}{r} = \frac{1}{re^{i\theta}}$$

$$= \frac{1}{z}$$

Recall

$z = re^{i\theta}$

Conclusion : $\frac{d}{dz}(\operatorname{Log} z) = \frac{1}{z}$

on D

That fits the real log in Calculus

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Important conventions we will use in the future:

$\ln x$: real log function

$\log z$: complex log function

$\log z = \ln|z| + i \arg z$

$\operatorname{Log} z$: principal value of complex log function

$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$

Remark:

- For all $z \neq 0$ in $\mathbb{C}$, we have

$$e^{\log z} = z \qquad z = re^{i\theta},\ \theta \in (-\pi, \pi]$$

Why?

$$e^{\log z} = e^{\ln|z| + i\arg z}$$

$$= e^{\ln r + i(\theta + 2n\pi)}$$

$$= e^{\ln r}\, e^{i(\theta + 2n\pi)}$$

$$= r e^{i\theta} \cancel{e^{i2n\pi}}$$

$$= re^{i\theta} = z$$

In above, $e^{\log z} = z$

• How about $\log(e^z)$? Does $\log(e^z) = z$?

Warning: $\log(e^z) \neq z$ in general

write $z = x + iy$

$$e^z = e^x e^{iy} \Rightarrow \begin{matrix} |e^z| = e^x \\ \theta = y + 2n\pi \end{matrix}$$

$$\log(e^z) = \ln|e^z| + i\arg(e^z)$$
$$= \ln e^x + i(y + 2n\pi)$$
$$= (x + iy) + i2n\pi = z + i2n\pi$$

• How about $\mathrm{Log}(e^z)$? Does $\mathrm{Log}(e^z) = z$?

Warning: $\mathrm{Log}(e^z) \neq z$ in general.