Recall:

Def^n: For $z \in \mathbb{C}$,

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

Remark: • $(\cos z)' = -\sin z$

$(\sin z)' = \cos z$

• If $z = x \in \mathbb{R}$,

the above $\cos z$, $\sin z$ coincide with the usual cos, sin functions in Calculus

Why? when $z = x \in \mathbb{R}$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{ix} + e^{-ix}}{2}$$

(Euler's ...) $$= \frac{\cos x + \cancel{i\sin x} + \cos x - \cancel{i\sin x}}{2}$$

$$= \cos x$$

Similarly, $\sin z = \sin x$

Q1: Find all $z \in \mathbb{C}$ such that $\sin z = 0$?

A: Recall $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

In Calculus
$\sin x = 0$
$x \in \mathbb{R}$
$\iff x = n\pi$
$n \in \mathbb{Z}$

Thus $\sin z = 0 \iff e^{iz} - e^{-iz} = 0$

$\iff e^{iz} = e^{-iz}$

(multiply by e^{iz}) $\iff e^{2iz} = e^{iz-iz} = e^{0}$
$= 1$

$\iff e^{2iz} = 1 = e^{i0}$

write $z = x + iy$

Then

$LHS = e^{2i(x+iy)} = e^{-2y+i(2x)} = \underbrace{e^{-2y}}_{r}\underbrace{e^{i2x}}_{e^{i\theta}}$

$RHS = \underbrace{1}_{r} \cdot \underbrace{e^{i0}}_{e^{i\theta}}$

$LHS = RHS$

$$\Longleftrightarrow \begin{cases} e^{-2y} = 1 \\ 2x = 0 + 2n\pi, \; n \in \mathbb{Z} \end{cases}$$

$$\Longleftrightarrow \begin{cases} -2y = \ln 1 = 0 \Longleftrightarrow y = 0 \\ x = n\pi, \; n \in \mathbb{Z} \end{cases}$$

Hence $\sin z = 0, \; z \in \mathbb{R}$

$$\Longleftrightarrow z = x + iy = n\pi + i0$$

$$= n\pi, n \in \mathbb{Z}$$

Q2: Find all $z \in \mathbb{C}$. s.t. $\cos z = 0$.

We have 2 ways

Way 1: Recall $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

Thus

$$\cos z = 0 \iff e^{iz} = -e^{-iz}$$

(multiple by e^{iz}) $\iff e^{2iz} = -1 = e^{i\pi}$

use polar form idea: Ex.

Way 2:

Claim: $\cos z = \sin(z + \frac{\pi}{2})$

Pf of claim: $LHS = \cos z = \frac{e^{iz} + e^{-iz}}{2}$

$$\boxed{\sin z = \frac{e^{iz} - e^{-iz}}{2i}}$$

$$\text{RHS} = \sin\Big(\underbrace{z+\frac{\pi}{2}}\Big)$$

(replace all "z" above by "$z+\frac{\pi}{2}$"

$$= \frac{e^{i(z+\frac{\pi}{2})} - e^{-i(z+\frac{\pi}{2})}}{2i}$$

$$= \frac{e^{iz}\cdot \cancel{i} - e^{-iz}(-\cancel{i})}{2\cancel{i}}$$

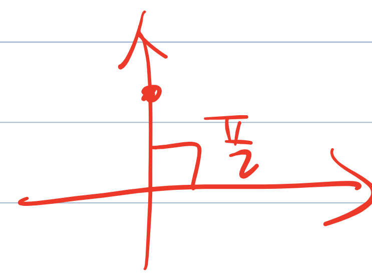
$$= \frac{e^{iz}+e^{-iz}}{2}$$

$$= \cos z$$

Hint:

$e^{i\frac{\pi}{2}} = i$

$e^{-i\frac{\pi}{2}} = -i$



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since $\cos z = \sin(\frac{\pi}{2}+z)$

$\Rightarrow \cos z = 0$ if and only if

$\sin(\frac{\pi}{2}+z) = 0$

This occurs if and only if $w = \frac{\pi}{2} + z = n\pi$

$$\left(\begin{aligned} &\Longleftrightarrow z = n\pi - \tfrac{\pi}{2},\ n \in \mathbb{Z} \quad (1) \\ &\Longleftrightarrow z = (n-1)\pi + \tfrac{\pi}{2},\ n \in \mathbb{Z} \quad (2) \end{aligned} \right)$$

Next, $\tan z$ $\Longleftrightarrow z = k\pi + \frac{\pi}{2},\ k \in \mathbb{Z}$ (3)

Defn: we define

$$\tan z = \frac{\sin z}{\cos z}, \quad \text{where } \cos z \neq 0$$

Recall $\cos z$, $\sin z$ are entire

(why? $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$, ...)

Q: where is $\tan z$ analytic?

Recall: If both f, g are analytic,

$\Rightarrow \frac{f}{g}$ is analytic except at $g = 0$.

A: $\tan z = \dfrac{\sin z}{\cos z}$ is analytic

at all pts except where $\cos z = 0$.

(i.e, analytic except $z = k\pi + \frac{\pi}{2},\ k \in \mathbb{Z}$)

Ex: prove $(\tan z)' = \sec^2 z$

Here $\sec z = \frac{1}{\cos z}$

Hint: use quotient rule for $\tan z = \frac{\sin z}{\cos z}$

Properties of trig function:

① $\sin^2 z + \cos^2 z = 1$

② $\sin(2z) = 2\sin z \cos z$

③ $\sec^2 z = \tan^2 z + 1$

④ $\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

⑤ $\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

Pf of ②:

$$\text{LHS} = \sin(2z) = \frac{e^{i2z} - e^{-i2z}}{2i}$$

$$\text{RHS} = 2\sin z\cos z$$

$$= \cancel{2}\,\frac{e^{iz} - e^{-iz}}{\cancel{2}i}\;\frac{e^{iz} + e^{-iz}}{2}$$

$$= \frac{(e^{iz})^2 - (e^{-iz})^2}{2i}$$

$$= \frac{e^{2iz} - e^{-2iz}}{2i} = \text{LHS}$$

Pf of ①: " $\sin^2 z + \cos^2 z = 1$ "

Hint: Need to compute

$$\left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 = 1$$

derivative

Calculus theory for $\mathbb{C}$: { Integral

Complex Integral and Contour

In Calculus, we defined real integral. $\int f(x)dx$

In complex analysis, we will define complex integral $\int \underline{f(z)dz}$

As a preparation; we will define today first something intermediate

① complex-valued functions of a real variable

$[a, b] \subseteq \mathbb{R}$

Consider $w: [a, b] \to \mathbb{C}$ $\qquad \underline{t \in [a, b]}$

Let $u(t) = \operatorname{Re} w(t)$; $v(t) = \operatorname{Im} w(t)$

So that $w(t) = u(t) + i v(t)$

We define $\underline{[a, b] \to \mathbb{C}}$

$u, v: [a, b] \to \mathbb{R}$

(1) Derivative

We define the derivative of w to be

$$w'(t) = \underline{u'(t)} + i\,\underline{v'(t)}$$

(2) Integral:

We define
$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

(3) Antiderivative $F : [a,b] \to \mathbb{C}$

Capital U

Let $F(t) = U(t) + i\,V(t)$ → Capital V

We say F is an antiderivative of $w(t) = u(t) + i\,v(t)$

$$\Longleftrightarrow \quad U'(t) = u(t), \quad V'(t) = v(t)$$

Notation: $F' = w$

Consequence of Fundamental Thm of Calculus:

If $F'(t) = w(t)$, then $\qquad \boxed{w = u + iv}$

$$\int_a^b w(t)\,dt = F(b) - F(a)$$

Pf: $\displaystyle \int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$

Recall: for $F = U + iV$,
$F' = w \iff U' = u,\ V' = v$

$\underline{\underline{\text{F.T.C}}}$ $\quad U(b) - U(a) + i\left(V(b) - V(a)\right)$

$$= \underbrace{\left(U(b) + iV(b)\right)}_{F(b)} - \underbrace{\left(U(a) + iV(a)\right)}_{F(a)}$$

$$= F(b) - F(a)$$

E.g: ① Fix $z_0 = x_0 + iy_0 \in \mathbb{C}$

$t \in (-\infty, \infty)$

E.X prove $\frac{d}{dt}(e^{z_0 t}) = (e^{z_0 t})'$

$= z_0 e^{z_0 t}$ $F' = w$

let $F(t) = e^{z_0 t} = e^{(x_0 + iy_0)t}$

$= e^{x_0 t}\cos(y_0 t) + i\, e^{x_0 t}\sin(y_0 t)$

$w(t) = z_0 e^{z_0 t} = (x_0 + iy_0)\, e^{(x_0 + iy_0)t}$

$= \cdots\cdots = u(t) + iv(t)$

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② $\int_0^1 (3it^2 + 2i)\, dt$ $\mathbb{R} \to \mathbb{C}$

$= \int_0^1 (\underbrace{0}_{u} + i\underbrace{(3t^2 + 2)}_{v})\, dt$

$= \cancel{\int_0^1 0\, dt}_{0} + i\int_0^1 (3t^2 + 2)\, dt$

$= i(t^3 + 2t)\Big|_0^1 = 3i$

③ $\int_0^{\frac{\pi}{2}} e^{it}\,dt$

we have 2 ways to do it

way 1 : (use definition)

$$\int_0^{\frac{\pi}{2}} e^{it}\,dt = \int_0^{\frac{\pi}{2}} (\underbrace{\cos t}_{u} + i\underbrace{\sin t}_{v})\,dt$$

$$= \int_0^{\frac{\pi}{2}} \cos t\,dt + i\int_0^{\frac{\pi}{2}} \sin t\,dt$$

$$= \cdots = 1 + i$$

E.X

way2 : $\left(\begin{array}{l} \text{use: If } F'(t) = w(t), \\ \int_a^b w(t)\,dt = F(b) - F(a) \end{array} \right)$

In " $\int_0^{\frac{\pi}{2}} e^{it}\,dt$ ",

$$w(t) = e^{it}$$

If $F(t) = \frac{e^{it}}{i} \Rightarrow$

$F'(t) = w(t)$

$\Rightarrow \int_0^{\frac{\pi}{2}} w(t)\,dt$

$= F(\frac{\pi}{2}) - F(0)$

$= \left.\frac{e^{it}}{i}\right|_{t=\frac{\pi}{2}} - \left.\frac{e^{it}}{i}\right|_{t=0}$

$= \frac{1}{i}\left(e^{i\frac{\pi}{2}} - e^{i0}\right) = \frac{1}{i}(i-1) = 1+i$

E.g ①.

$\frac{d}{dz}(e^{z_0 t})$

$= z_0 e^{z_0 t}$

Hint:

$(e^{it})' = ie^{it}$

$\left(\frac{e^{it}}{i}\right)' = e^{it}$

Recall Calculus:

$$\underbrace{\left| \int_a^b f(x)\,dx \right|}_{\text{Cancellation possible}} \leq \underbrace{\int_a^b |f(x)|\,dx}_{\text{no Cancellation}}$$

Similarly, for $w(t): [a, b] \to \mathbb{C}$

$$\underbrace{\left| \int_a^b w(t)\,dt \right|}_{\text{possible cancellation}} \leq \underbrace{\int_a^b |w(t)|\,dt}_{\text{no cancellation}}$$

Standard Identities: