Recall: • $\mathbb{C}$ = the set of complex numbers

• every $z \in \mathbb{C}$ can be written as

$z = x + iy$, where $x, y \in \mathbb{R}$.

Note: $x = \mathrm{Re}(z)$

$y = \mathrm{Im}(z)$

• Algebra in $\mathbb{C}$:

① Addition:

$$(a+ib) + (c+id) = (a+c) + i(b+d)$$

② Multiplication:

$$(a+ib)\cdot(c+id) = (ac-bd) + i(bc+ad)$$

Note: distribution law still holds

③ Substraction

$$(a+ib) - (c+id) = (a-c) + i(b-d)$$

④ Division

Note

- $\frac{c+id}{a+ib}$ is only defined when $a+ib \neq 0$

$\Leftrightarrow$ a, b are not both 0

- In this case, $\frac{c+id}{a+ib} = (c+id) \cdot (a+ib)^{-1}$

Q: what is $(a+ib)^{-1}$?

A: $(a+ib)^{-1}$ = the number whose product with $a+ib$ equals 1

Given $a+ib \neq 0$

Fact: $\boxed{(a+ib)^{-1} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}}$ (*)

Why? we will give 2 proofs.

Proof 1: check their product $= 1$.

$$(a+ib)\left(\frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}\right) = \cdots\cdots$$

distribution law

$$= \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2} = 1$$

Proof 2:

$$(a+ib)^{-1} = \frac{1}{a+ib} = \frac{1}{a+ib}\,\frac{(a-ib)}{(a-ib)}$$

$$= \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

Remark: If $z = a+ib$, then $a-ib$ is called the conjugate of z

Notation: $\bar{z} = a-ib$

" $\bar{z}$ " reads " z bar "

E.g 1: compute $\frac{2+i}{1+2i}$ and $Re\left(\frac{2+i}{1+2i}\right)$

2 ways

way 1: $\frac{2+i}{1+2i} = (2+i)(1+2i)^{-1}$

then use the formula (*)

for $(a+ib)^{-1}$

way 2: (use "conjugate of denominator")

$$\frac{2+i}{1+2i} = \frac{2+i}{1+2i}\,\frac{1-2i}{1-2i} = \frac{(2+i)(1-2i)}{1^2+2^2}$$

$$= \frac{2-4i+i+\overset{2}{\cancel{i(-2i)}}}{5} = \frac{4-3i}{5}$$

$\Rightarrow Re\left(\frac{2+i}{1+2i}\right) = \frac{4}{5}$

Remark: The complex numbers form a field

with "+", "×"

"−", "÷"

a concept from abstract algebra.

Some properties of conjugates:

(1) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

(2) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

Pf: (1) write $z_1 = a+ib$

$z_2 = c+id$

then $z_1+z_2 = (a+c)+i(b+d)$

$LHS = \overline{z_1+z_2} = (a+c)-i(b+d)$

We also have $\bar{z}_1 = a-ib$

$\bar{z}_2 = c-id$

$\Rightarrow RHS = \bar{z}_1 + \bar{z}_2 = (a+c)-i(b+d)$

Hence $LHS = RHS$

and (1) holds.

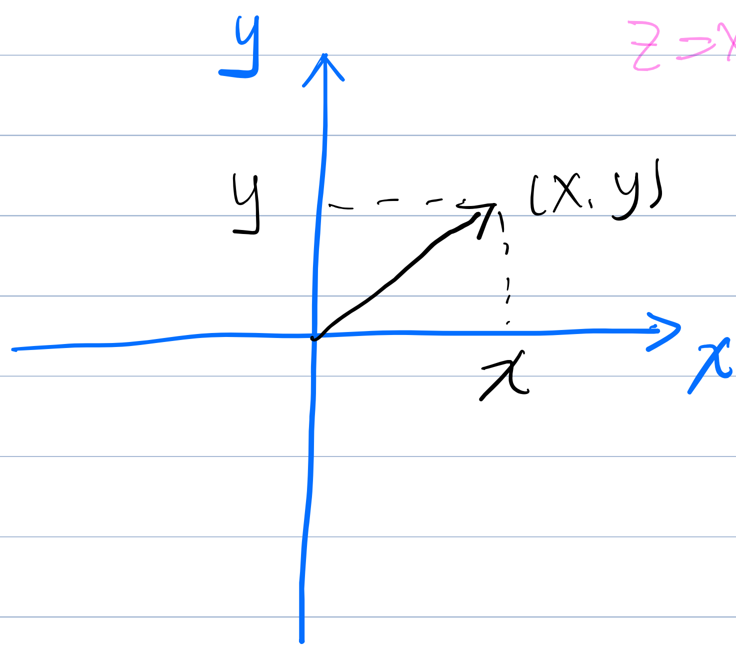
(2) Exercise.

Geometric meaning of complex numbers

If $z = x + iy$, where $x, y \in \mathbb{R}$,

then we identify z as a vector in xy-plane

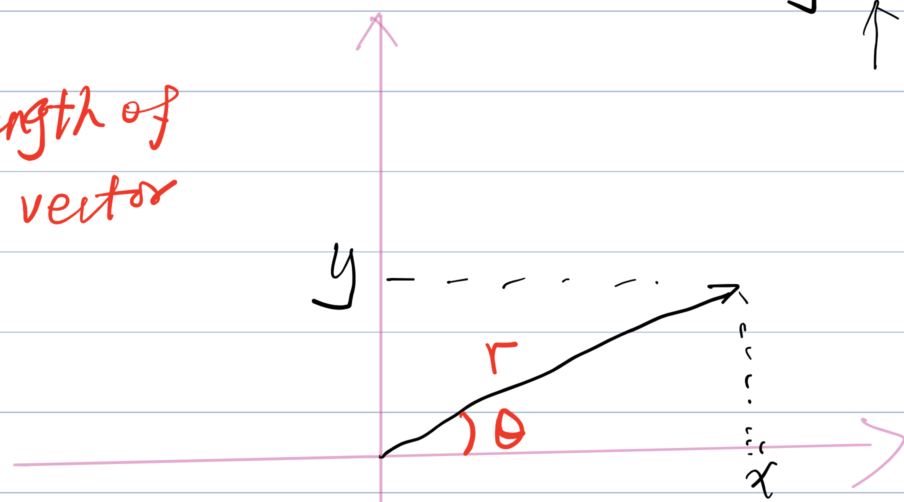
$z = x + iy \sim (x, y)$



Exponential/polar form

$x+iy \sim (x,y)$

r: length of the vector



Trig:

$$r = \sqrt{x^2+y^2}$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$

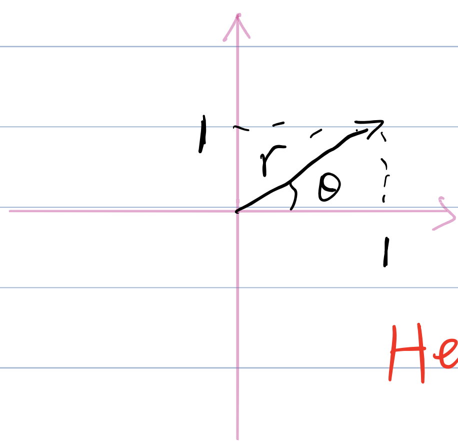
We can then write z in terms of r, θ

$$\begin{aligned} z &= x+iy \\ &= r\cos\theta + ir\sin\theta \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$

Note: we can add any integer multiple of 2π to θ, because

$$\cos\theta = \cos(\theta + 2k\pi), \quad k \in \mathbb{Z}$$
$$\sin\theta = \sin(\theta + 2k\pi)$$

E.g: Let $z = 1 + i \sim (1,1)$



$\theta = \frac{\pi}{4}$

or $\frac{\pi}{4} + 2\pi$

or $\frac{\pi}{4} + 2k\pi$

$r = \sqrt{1^2 + 1^2} = \sqrt{2}$

Hence

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

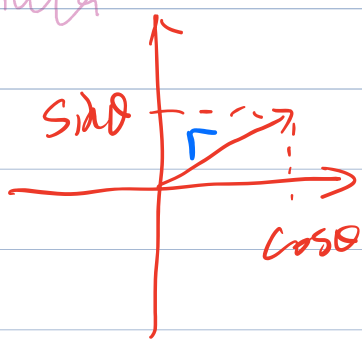
$$= \sqrt{2}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + i\sin\left(\frac{\pi}{4} + 2k\pi\right)\right)$$

Note: We will use the expression "$\cos\theta + i\sin\theta$" very often

Important formula:

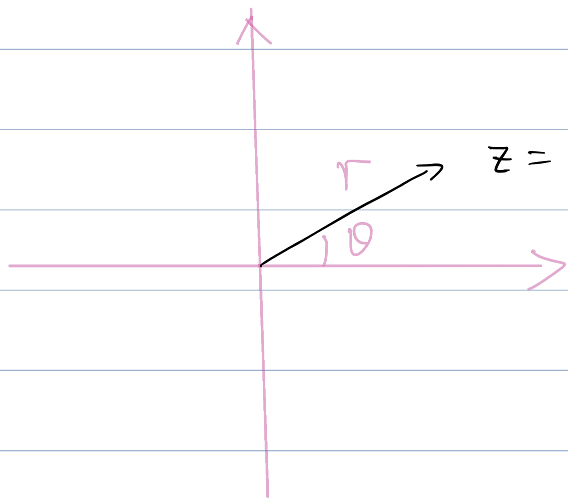
Euler formula → $$e^{i\theta} \triangleq \cos\theta + i\sin\theta$$

$$(\cos\theta, \sin\theta)$$



$$r = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

Summary:



$z = x+iy \leftrightarrow (x, y) \leftrightarrow r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$

Two ways to write z:

① $z = x+iy$ ← rectangular form of z

② $z = re^{i\theta}$ ← exponential/polar form of z 'modulus' = 'length'

Def^n: For $z = x+iy = re^{i\theta}$

① $r = \sqrt{x^2+y^2}$ is called the modulus of z

Notation: $|z| = r = \sqrt{x^2+y^2}$

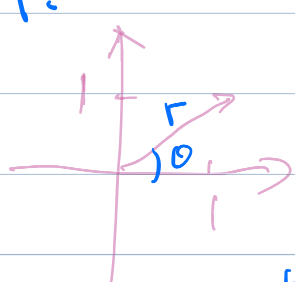
② θ is called the argument of z

It is defined up to $2k\pi$, $k \in \mathbb{Z}$

Notation: $\arg z = \theta$

E.g: Let $z = 1+i$. Find $|z|$, $\arg z$

A:



$\theta = \frac{\pi}{4}$ or $\frac{\pi}{4} + 2k\pi$

$r = \sqrt{2}$

$z = \sqrt{2}\, e^{i(\frac{\pi}{4} + 2k\pi)}$, $k \in \mathbb{Z}$

Hence $|z| = \sqrt{2}$, $\arg z = \frac{\pi}{4} + 2k\pi$, $k \in \mathbb{Z}$

Defn: Every $z \neq 0$ always has a unique argument θ in the range

$$-\pi < \theta \leq \pi$$

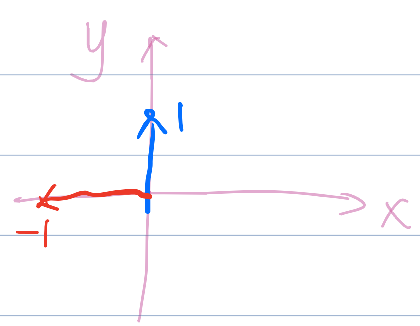
we can this special θ to be the principal argument of z

Notation: $\operatorname{Arg} z$

E.g:

$\operatorname{Arg}(i) = \frac{\pi}{2}$, $\arg(i) = \frac{\pi}{2} + 2k\pi$

$\operatorname{Arg}(-1) = \pi$, $\arg(-1) = \pi + 2k\pi$



$i = 0 + 1 \cdot i \sim (0, 1)$

$-1 = -1 + 0 \cdot i \sim (-1, 0)$