Recall:

Complex integral (a.k.a. Contour integral)

Def^n: Let $C: z(t), a \le t \le b$ be a contour

Let $f(z)$ be a function defined on C and continuous on C

we define the contour integral

$$\int_C f(z)dz \triangleq \int_a^b f(z(t))\, z'(t)\, dt$$

How to memorize it?

$$dz = \frac{dz}{dt}\, dt = z'(t)\, dt$$

$$\Rightarrow \int_C f(z)\, dz = \underbrace{\int_a^b}_{z(t),\ a \le t \le b} f(z(t))\, z'(t)\, dt$$

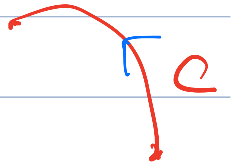
$z \in C$

Important Remark:

In general, the integral

$\int_C f(z)dz$ does NOT depend on the parametrization of C (need to keep orientation)

More precisely,



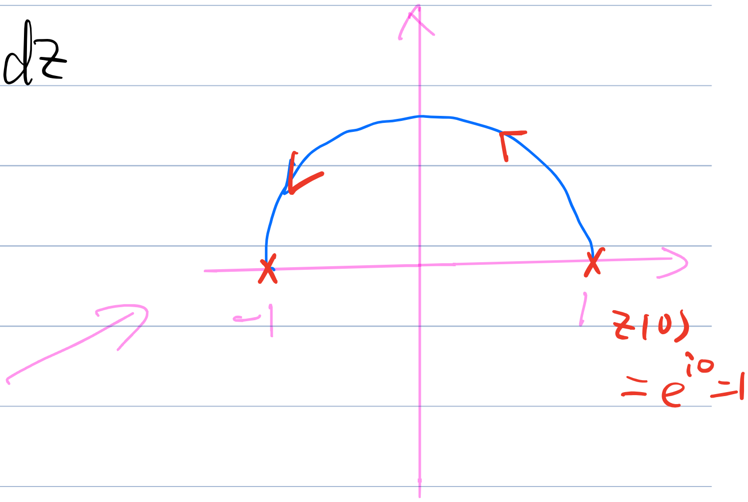
If $\begin{cases} z_1(t), & a_1 \le t \le b_1 \quad ① \\ z_2(t), & a_2 \le t \le b_2 \quad ② \end{cases}$

parametrize the same C, and give the same orientation. then the computations of $\int_C f(z)dz$ using ①, ② give the same answer.

Recall we did last time:

E.g: let C: semi-circle as below, counterclockwise

Compute $\int_C z \, dz$



We used 3 different parametrizations of C:

① $z(t) = e^{it}$, $0 \le t \le \pi$

② $z(t) = e^{2it}$, $0 \le t \le \frac{\pi}{2}$

③ $z(t) = e^{3it}$, $0 \le t \le \pi/3$

We got the same answer in ①, ②, ③.

$$\int_C z\,dz = 0.$$

Remark: In the future, if a question only gives the contour C and its orientation (without providing the precise parametrization), then you can always choose any parametrization you like to compute $\int_C f(z)\,dz$

↓

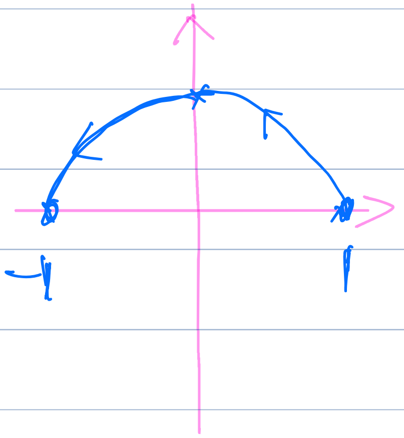
Tips: pick the simpliest one

E.g ① Compute $\int_C \bar{z}\, dz$

← radius = 1

C: is the semi-circle in the graph.

counter-clockwize oriented



Take any parametrization of C as you like:

$z(t) = e^{it}, \quad 0 \le t \le \pi$

By defn ($\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dt$)

$$\int_C \bar{z}\, dz = \int_0^{\pi} \overline{z(t)}\, z'(t)\, dt$$

$$= \int_0^{\pi} \overline{e^{it}}\; i e^{it}\, dt$$

$$\boxed{\frac{d}{dt} e^{it} = i e^{it}}$$

$$= \int_0^{\pi} \cancel{e^{-it}}\; i \cancel{e^{it}}\, dt$$

$$\boxed{\overline{e^{it}} = e^{-it}}$$

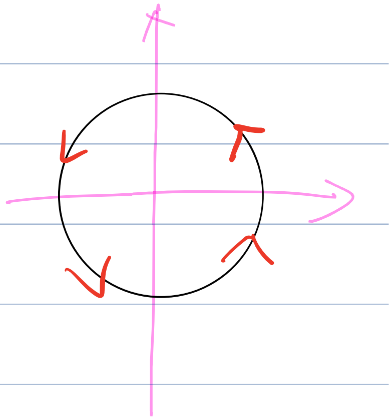
$$= i \int_0^{\pi} dt = i\pi .$$

$$\overline{e^{it}} = \overline{\cos t + i\sin t} = \cos t \overset{\vee}{-} i\sin t$$

② C: unit circle $\{|z|=1\}$

counter clock-wise

Compute $\int_C \frac{1}{z}\,dz$



Step1:

A: Take a parametrization of C.

$$z(t) = e^{it}, \quad 0 \le t \le 2\pi$$

Step2: (Use def^n to compute)

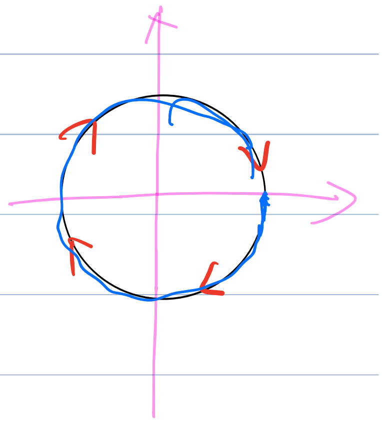
$$\int_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{z(t)}\, z'(t)\,dt$$

$$= \int_0^{2\pi} \frac{1}{\cancel{e^{it}}}\, i\,\cancel{e^{it}}\,dt$$

$$= \int_0^{2\pi} i \, dt = 2\pi i$$

③ γ: circle $\{|z| = 1\}$ clockwise

Compute $\int_\gamma \frac{1}{z} dz$



A: $\arg = -t$

Step 1: Take a parametrization.

$$z(t) = \underline{e^{\boxed{-it}}}, \quad 0 \le t \le 2\pi$$

$$\Downarrow$$

Step 2:

$$z'(t) = -i e^{-it}$$

$$\int_\gamma \frac{1}{z}\,dz$$

$$= \int_0^{2\pi} \frac{1}{z(t)}\, z'(t)\,dt$$

$$= \int_0^{2\pi} \frac{1}{\cancel{e^{-it}}}\,(-i\cancel{e^{-it}})\,dt$$

$$= \int_0^{2\pi} (-i)\,dt$$

$$= -2\pi i$$

Remark: Note in the above,

$$\gamma = -C$$

(same curve opposite orientation)

$$\int_\gamma \frac{1}{z}\,dz = -\int_C \frac{1}{z}\,dz$$

In general, for any contour C

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$$

Upper bound for moduli of contour integrals

Lemma: Let C be a contour. Assume

- $\boxed{|f(z)| \le M}$, $\forall z \in C$

(Here M is a positive number)

- length$(C) = L$

Then $$\left| \int_C f(z)\,dz \right| \le M \cdot L$$

pf: (NOT required)

$$\left| \int_C f(z)\,dz \right|$$

‖ defn.

ψ

$$= \left| \int_a^b f(z(t))\, z'(t)\, dt \right|$$

$$\le \int_a^b |f(z(t))|\,|z'(t)|\,dt$$

$$\le M \int_a^b |z'(t)|\,dt$$

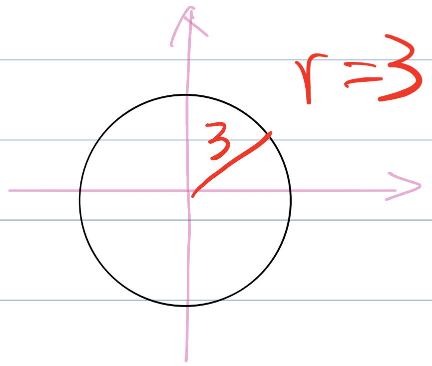
$$\le M \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

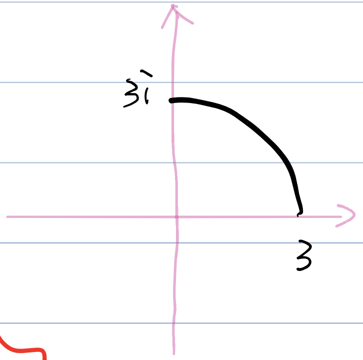
$$\le M \text{ length}(C)$$

$$\le ML$$

E.g. let C: $z(t) = 3e^{it}$ $\quad 0 \le t \le \pi/2$

① Find length(C) $= \frac{3\pi}{2} = L$





Length $= 2\pi \cdot r = 6\pi$

length(C) $= \frac{6\pi}{4}$
$= \frac{3\pi}{2}$

② Find a bound for $\left|\int_C (\bar{z}^2 + i)\, dz\right|$

A: For $z \in C$, $z = 3e^{it} \Rightarrow |z| = 3$

$\Rightarrow \quad |\bar{z}^2| = |\bar{z}|^2 = |z|^2 = 9$

Hence
$|f(z)| = |\bar{z}^2 + i| \le |\bar{z}^2| + |i|$

$$\le 9 + 1 = \underset{\underset{M}{\uparrow}}{10}$$

By lemma,

$$\left|\int_C f(z)\,dz\right| \le M\cdot L = 10\cdot \frac{3\pi}{2}$$
$$= 15\pi$$

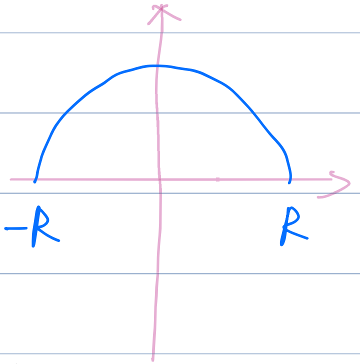
E.g. Let

$$C_R: \quad z(t) = Re^{it}, \quad 0 \le t \le \pi$$

with $R > 1$

prove $$\lim_{R\to\infty} \int_{C_R} \frac{z-2}{z^4+1}\,dz = 0$$

$C_R: \; z(t) = Re^{it}$
$0 \le t \le \pi$



Pf: Let $$I_R \triangleq \int_{C_R} \frac{z-2}{z^4+1}\,dz$$

Goal: show $I_R \to 0$ as $R \to \infty$

· Note :

$$\text{length}(C_R) = \frac{2\pi R}{2} = \boxed{\pi R} \leftarrow L$$

$$\left(f(z) = \frac{z-2}{z^4+1} \right)$$

For $z \in C_R$, $\Rightarrow$ $|z| = R$

$$|z-2| \underset{\uparrow}{\leq} |z| + 2 = R+2 > 0$$

trig. ineq.

$$|z^4+1| \underset{\uparrow}{\geq} |z|^4 - 1 = R^4 - 1 > 0$$

reverse trig. ineq.

$$(|z+w| \geq |z| - |w|)$$

Hence

$$|f(z)| = \left| \frac{z-2}{z^4+1} \right| \leq \boxed{\frac{R+2}{R^4-1}}$$

M

By lemma.

$$|I_R| = \left| \int_{C_R} f(z)\, dz \right|$$

$$\le M \cdot L = \underline{\frac{R+2}{R^4-1} \cdot \pi R}$$

Note:

$$\underset{\downarrow}{0} \le |I_R| \le \pi \frac{R^2+2R}{R^4-1} \underset{\uparrow}{\le} \pi \underbrace{\frac{\boxed{\frac{R^2}{R^4}}^{\to 0} + \boxed{\frac{2R}{R^4}}^{\to 0}}{\cancel{\frac{R^4}{R^4}}_{1} - \boxed{\frac{1}{R^4}}^{\to 0}}}_{\downarrow}$$

$$0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0$$

divide top. bottom by R^4

By squeeze Thm,

$$\lim_{R\to\infty} |I_R| = 0$$

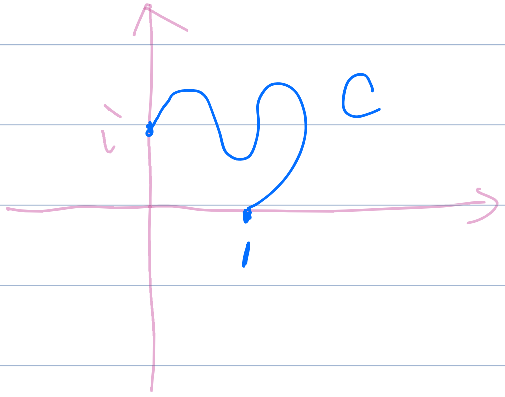
$$\Rightarrow \lim_{R\to\infty} I_R = 0$$

If $|a_n| \to 0 \iff a_n \to 0$

Antiderivative and path Independence

Lemma: let $f(z)$

E.g. let C be as below.



① $\int_C z \, dz$

② $\int_C (z^2 + i) \, dz$

Pf of lemma :