Recall:

Method of Anti-derivative to compute $\int_C f(z)\,dz$:

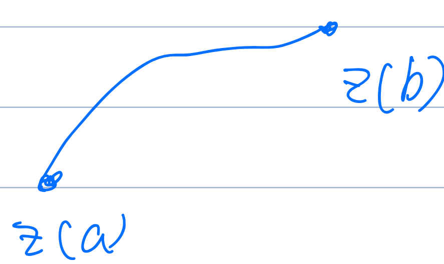
Tips on how to use it:

Find F, D s.t

① $F' = f$ on D

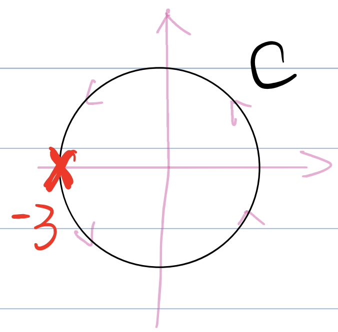
② The contour C is entirely in D.

$$\Rightarrow \quad \int_C f(z)\,dz = F\Big|_{z(a)}^{z(b)}$$



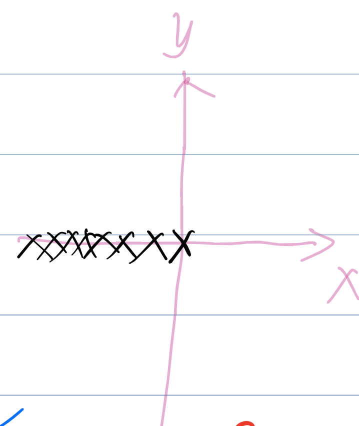
② Let C be as in ①:

$C: z(t) = 3e^{it}, \quad 0 \le t \le 2\pi$ ← Closed



Compute $\int_C \frac{1}{z}\,dz$

Warning: Although $(\operatorname{Log} z)' = \frac{1}{z}$, but



$\operatorname{Log} z$ is only analytic on

$\Omega = \{re^{i\theta} \mid r>0, -\pi<\theta<\pi\}$

and $(\operatorname{Log} z)' = \frac{1}{z}$ on Ω

$\Omega = \mathbb{C} - \{0, \text{negative } x\text{-axis}\}$

Note: Ω misses one pt of C.

thus C is NOT entirely contained in Ω.

Conclusion:

We will not use the method of anti-derivative!

Instead, we compute it by defn!

$$\int_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{z(t)}\, z'(t)\,dt$$

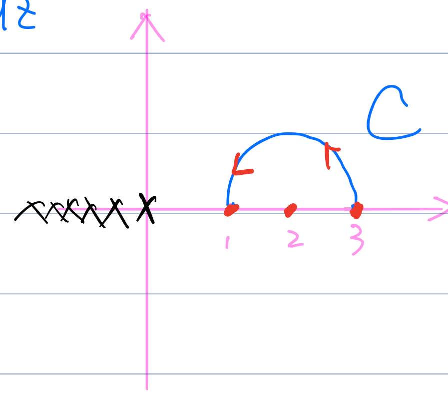
$(z(t) = 3e^{it})$

$$= \int_0^{2\pi} \frac{1}{\cancel{3e^{it}}}\, \cancel{3} i \cancel{e^{it}}\,dt = \int_0^{2\pi} i\,dt$$

$$= 2\pi i$$

③ $C: z(t) = 2 + e^{it}, \quad 0 \le t \le \pi$

Compute $\int_C \frac{1}{z}\, dz$



A:

Take $F = \operatorname{Log} z$

and $D = \Omega = \{ re^{i\theta} \mid r > 0, \ -\pi < \theta < \pi \}$

$\left(= \mathbb{C} - \{0, \text{negative } x\text{-axis}\} \right)$

Then ① F is analytic

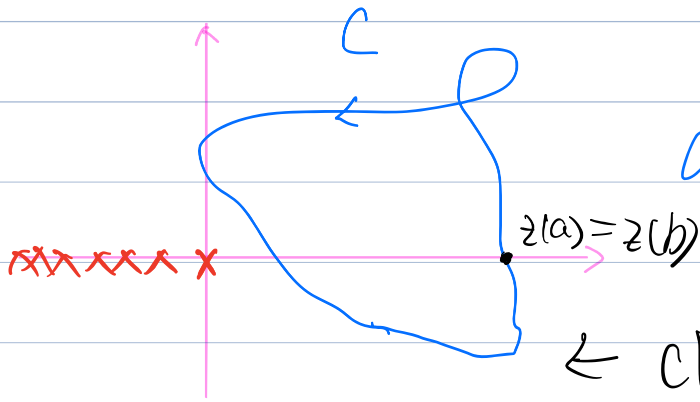
and $F' = \frac{1}{z}$ in D

② C is entirely in D.

Hence $\int_C \frac{1}{z}\, dz = \operatorname{Log} z \Big|_{z(a)=3}^{z(b)=1}$

$$= \operatorname{Log} 1 - \operatorname{Log} 3$$

④ $$= \ln 1 - \ln 3 = -\ln 3$$



Compute $\int_C \frac{1}{z}\,dz$

← closed contour

A: Take $F = \operatorname{Log} z$

$D = \mathbb{C} - \{0, \text{ negative } x\text{-axis}\}$

Note: ① $F' = \frac{1}{z}$ on D

② C is entirely in D

$$\Rightarrow \int_C \frac{1}{z}\,dz = \operatorname{Log} z \Big|_{z(a)}^{z(b)} = 0$$

Remark: For $\int_C \frac{1}{z}\,dz$,

if C is closed $(z(a)=z(b))$ and lies entirely in

$$D = \mathbb{C} - \{0, \text{ negative } x\text{-axis}\}$$

$$\Rightarrow \quad \int_C \frac{1}{z}\,dz = 0$$

Recall:

Lemma:

If f has an anti-derivative F on D,

then $\int_C f(z) = 0$ for all closed contour C lying entirely in D

Q: What about the converse of the lemma ?

That is,

(i) f admits an anti-derivative F in D

by Lemma $\Downarrow$ $\Uparrow$?

(ii) $\int_C f(z)\,dz = 0$ for all closed contour C lying entirely in D

A: Yes!

Thm: (i) $\Longleftrightarrow$ (ii)

Pf: Read P146 of the book.

for the direction "(ii) $\Rightarrow$ (i)".

Summary:

Now we have two ways to compute $\int_C f(z)\,dz$:

① by def^n

② by using anti-derivative.

Next, we will discuss a 3rd way to compute $\int_C f(z)\,dz$:

③ To use Cauchy-Goursat Theorem.

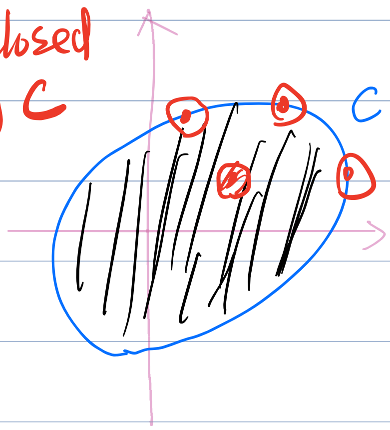
no self-intersection except $z(a) = z(b)$

Cauchy - Goursat Thm (C.-G. Thm)

Let C: simply closed contour

If $f(z)$ is analytic at all points interior to C and at all points on C

enclosed by C



then $\int_C f(z)\,dz = 0$.

Tips to use C.-G. Thm

we verify:

① C is simple closed

② f: analytic at all points enclosed by C

③ f: analytic at pts on C

Remark: Orientation of C does NOT matter

Pf: will do it later.

E.g: $\int_C e^{z^2} \cos(z^5)\, dz$

where C: any simply closed contour in $\mathbb{C}$

A: ① C is simply closed

②, ③: Note $f = e^{z^2} \cos(z^5)$

e^{z^2} ← entire　　　　$\cos(z^5)$ → entire

$\underbrace{e^w}_{\text{entire}}$　$\underbrace{w = z^2}_{\text{entire}}$　　　　$\underbrace{\cos w}_{\checkmark}$　$\underbrace{w = z^5}_{\checkmark}$

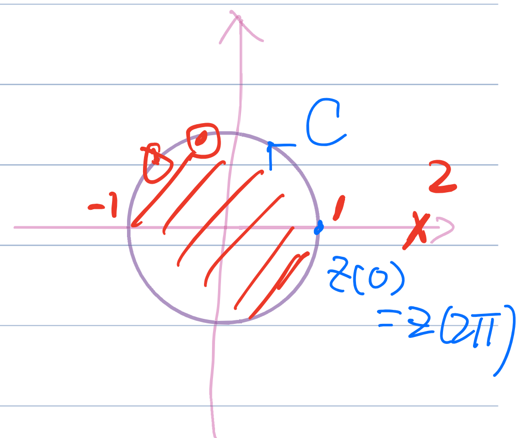
$\Rightarrow e^{z^2}, \cos(z^5)$ are entire (analytic on $\mathbb{C}$)

Hence $f(z) = e^{z^2} \cdot \cos(z^5)$ is entire

By C.-G. Thm. $\int_C f(z)\,dz = 0$.

E.g: let $C: z(t) = e^{it}, \quad 0 \le t \le 2\pi$

compute $$\int_C \frac{z^6 + 2z^2 + 100}{z-2}\,dz$$



A: (Idea: Use. C.-G. Thm)

We verify

① C: simply closed

②, ③: let $f(z) = \dfrac{z^6 + 2z^2 + 100}{z-2}$

$\Rightarrow$ f: analytic at $\mathbb{C} - \{2\}$.

$\Rightarrow$ f: analytic at all pts interior to C and on C

By C.-G. Thm. $\Rightarrow \int_C f(z)\,dz = 0$.

Pf of C.-G. Thm: (NOT required)

Let's first recall

C.-G. Thm.

If $f(z)$: analytic at all points interior to and on a simple closed contour C, then

$$\int_C f(z)\,dz = 0.$$

Pf: write $f(z) = u(x,y) + iv(x,y)$

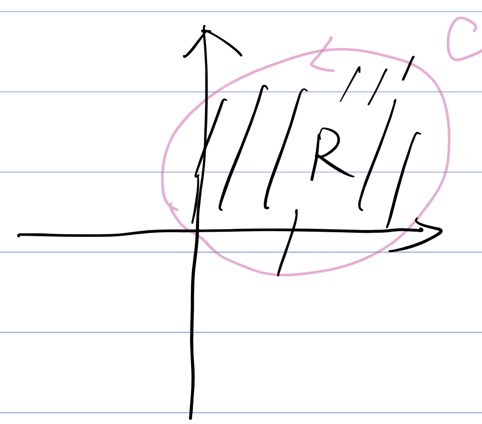
We will give a proof of C.-G. Thm under the additional assumption that u_x, u_y, v_x, v_y are continuous.

Recall Calculus: Green's Thm.

Let $P(x,y), Q(x,y): \mathbb{R}^2 \to \mathbb{R}$ be functions with continuous first order derivatives (meaning P_x, P_y, Q_x, Q_y are continuous) $\Longrightarrow$

$$\int_C P\,dx + Q\,dy = \iint_R (Q_x - P_y)\,dx\,dy$$

counterclockwize



Recall $z = x+iy \Rightarrow dz = dx + i\,dy$

Recall $f = u + iv$ analytic

Then

$$\int_C f(z)\,dz = \int_C (u+iv)(dx + i\,dy)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

$$= \iint_R (\underbrace{-v_x - u_y}_{=0})\,dx\,dy + i\iint_R (\underbrace{u_x - v_y}_{=0})\,dx\,dy$$

$$= 0$$

Def^n: (simply connected domains)

Let $D \subseteq \mathbb{C}$ be a domain. ← open + connected

We say D is a simply connected domain

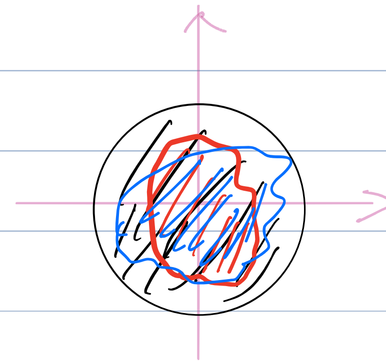
$\Updownarrow$

every simply closed contour in D → no self-intersections (except $z(a) = z(b)$)

encloses only points in D

E.g. Are the following domains simply connected.

① $D = \{|z| < 1\}$

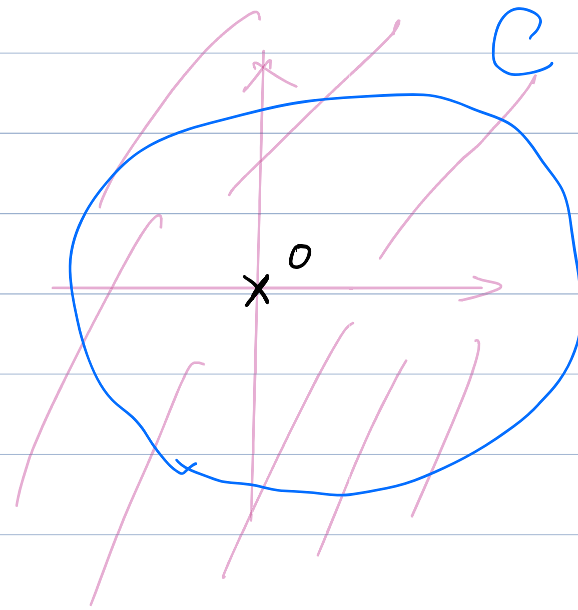


Yes, simply-connected

② $D = \mathbb{C}$

Yes, simply-connected

③ $D = \mathbb{C} - \{0\}$



In this pic,

C is simple closed in D

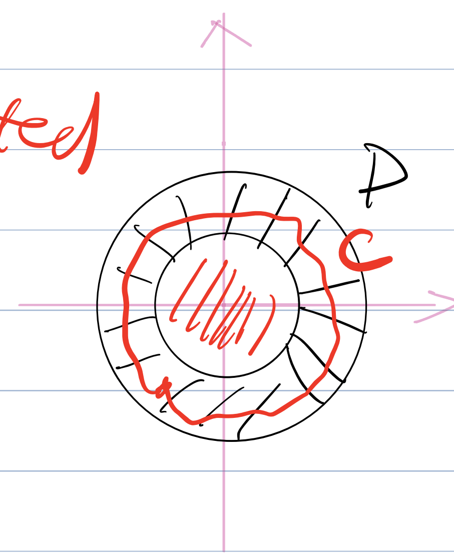
but C encloses 0

and $0 \notin D$!

No, not simply connected

④ $D = \{1 < |z| < 2\}$

No, not simply connected

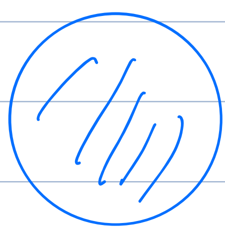


Remark: Intuitively

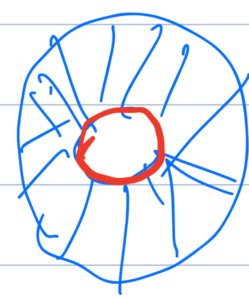
D is simply connected

$\Updownarrow$

D has no "holes"



simply-connected



NOT simply connected.