Recall:

Def^n: (simply connected domains)

Let $D \subseteq \mathbb{C}$ be a domain.

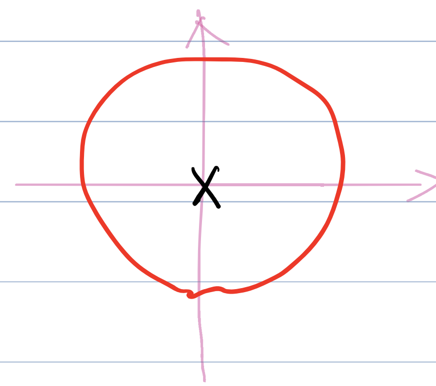
We say D is a simply connected domain

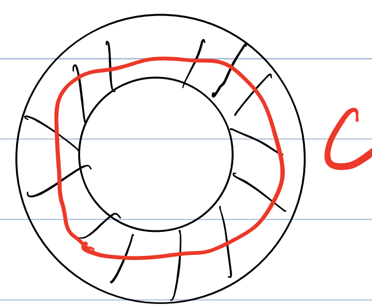
$\Updownarrow$

every simply closed contour in D
encloses only points in D

E.g

Simply connected domains	non-simply connected domains
① $\{\|z\| < 1\}$ disks	① $\mathbb{C} - \{0\}$
② $\mathbb{C}$	② $\{1 < \|z\| < 2\}$



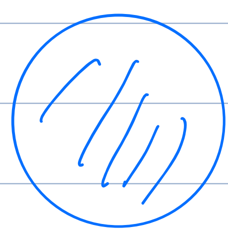


Remark: Intuitively

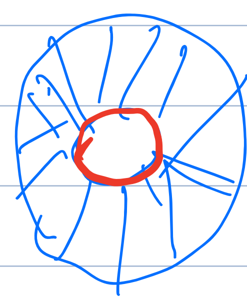
D is simply connected

$\Updownarrow$

D has no "holes"



simply-connected



NOT simply connected.

Thm: Let

D: simply connected domain in $\mathbb{C}$

C: closed contour in D

$z(a) = z(b)$

f: analytic in D

Then $\displaystyle\int_C f(z)\,dz = 0$

Pf: (NOT required)

self-intersection

no self ... except $z(a) = z(b)$

① If C is simply closed, then applying Cauchy - Goursat Thm.

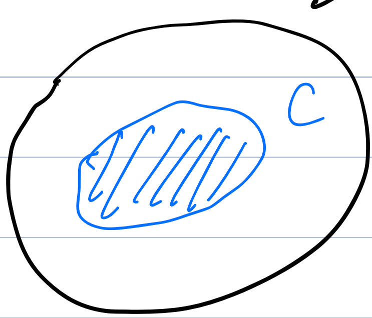
Recall : C. - G. Thm :

Let C : simply closed contour

f : analytic at pts interior to C and on C

$$\Rightarrow \int_C f(z)\,dz = 0$$

D simply connected



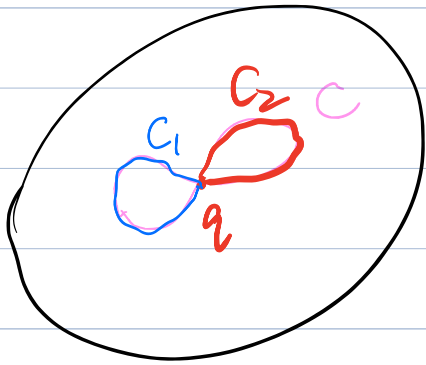
f : analytic at pts enclosed by C and on C

$$\Rightarrow \int_C f(z)\,dz = 0$$

$$\int_C f(z)\,dz = 0$$

② If C self-intersects once, then

C is NOT simply-closed



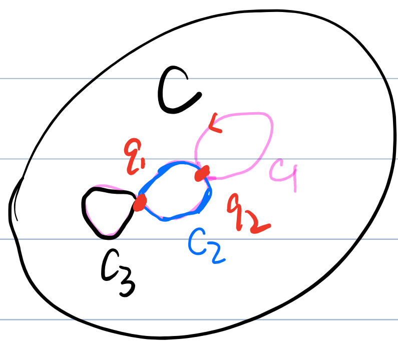
Note: $C = C_1 \cup C_2$

C_1 and C_2 are simply closed.

$$\Rightarrow \int_C f(z)\,dz = \underbrace{\int_{C_1} f(z)\,dz}_{\overset{\text{C.-G. Thm}}{=}\, 0} + \underbrace{\int_{C_2} f(z)\,dz}_{=\,0}$$

not simply closed.

③ If C self-intersects twice, then



$$\int_C f(z)\,dz = \underbrace{\int_{C_1} f(z)\,dz}_{=0} + \underbrace{\int_{C_2} f(z)\,dz}_{=0} + \underbrace{\int_{C_3} f(z)\,dz}_{=0} = 0$$

C-G. Thm   C-G. Thm

$C = C_1 \cup C_2 \cup C_3$

C_1, C_2, C_3 simply closed

In general cases, the idea is similar.

Let's repeat what just proved:

Thm: Let

D: simply connected domain

$C \subseteq D$: any closed contour

f: analytic in D

Then $\int_C f(z)\,dz = 0$

Also recall in last lecture, we have

Thm:

(i) f admits an anti-derivative F in D

$\Updownarrow$

(ii) $\int_C f(z)\,dz = 0$ for all closed contour C lying entirely in D

Hence we obtain:

Thm: Any analytic f on a simply connected domain D always admits an antiderivative F on D ($F' = f$ on D)

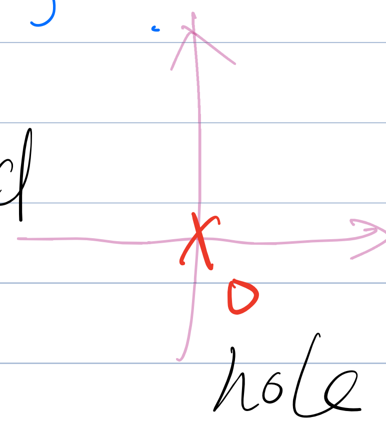
Q: What happens if D is not simply connected?

A: If D is not simply connected, then it depends:

some analytic f on D has an antiderivative on D, some does not.

Eg: $D = \mathbb{C} - \{0\}$

not simply connected



① let $f_1 = z$

Then take $F_1 = \frac{z^2}{2} \Rightarrow F_1' = f_1$

② let $f_2 = \frac{1}{z}$: analytic on D

(Although $(\text{Log}\, z)' = \frac{1}{z}$, $\text{Log}\, z$ is NOT analytic on D)

Indeed, $f_2 = \frac{1}{z}$ has no anti-derivative on D.

Cauchy Integral formula

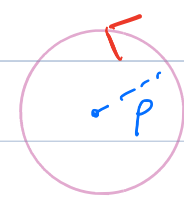
preparation:

We first recall the following example:

Recall:

Q: let $C: z(t) = \rho e^{it}$, $0 \le t \le 2\pi$, $\rho > 0$

compute $\int_C \frac{1}{z}\,dz$



A: $\int_C \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{z(t)} z'(t)\,dt$

$= \int_0^{2\pi} \frac{1}{\cancel{\rho}\cancel{e^{it}}} \cancel{\rho}\, i\, \cancel{e^{it}}\,dt$

$= \int_0^{2\pi} i\,dt = 2\pi i$

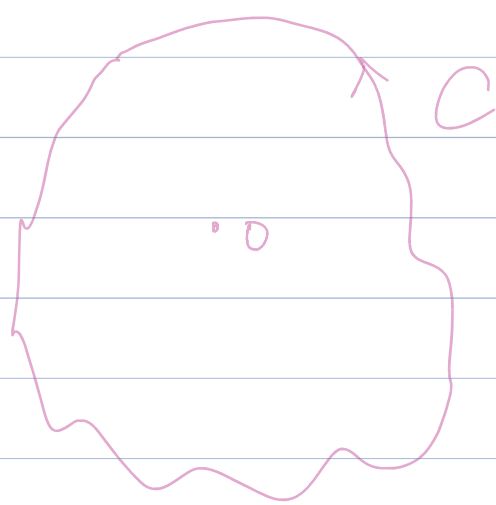
Note: The result does not depend on ρ

Now:

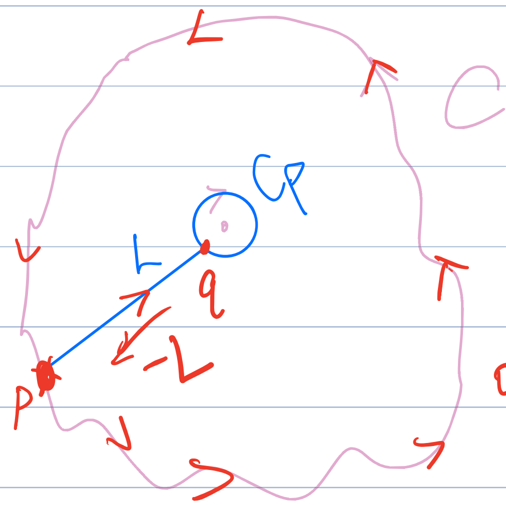
Q: Let C be a simply closed contour with $z=0$ enclosed. (positively oriented)

In the future,

- "+ oriented" $\Leftrightarrow$ counterclockwise
- "− oriented" $\Leftrightarrow$ clockwise



A: (pf NOT required)



do a "key hole surgery" to C

our travel starts with p, and ends with p

Intuitive Idea:

Let $\gamma := \overset{(1)}{C} + \overset{(2)}{L} - \overset{(3)}{C_p} - \overset{(4)}{L}$ ← closed contour

By C.-G. Thm.

$$\int_\gamma f(z)\,dz = 0$$

$$\Rightarrow \int_C f(z)\,dz + \cancel{\int_L f(z)\,dz} + \int_{-C_p} f(z)\,dz + \int_{-L} f(z)\,dz = 0$$

$$\|$$

$$\cancel{-\int_L f(z)\,dz}$$

$$\Rightarrow \int_C f(z)\,dz + \int_{-C_\rho} f(z)\,dz = 0$$

$$\|$$

$$-\int_{C_\rho} f(z)\,dz$$

$$\Rightarrow \int_C f(z)\,dz = \int_{C_\rho} f(z)\,dz = 2\pi i$$

In general, we have:

Thm: (principle of deformation of paths)

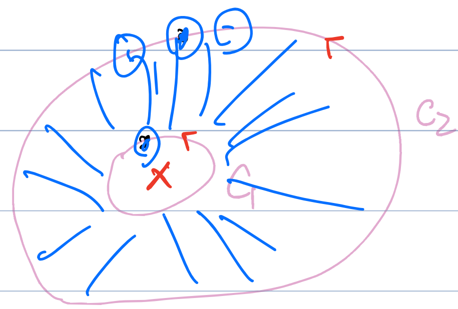
Let

- C_1, C_2: positively oriented, simply closed

  C_1 is inside C_2

- f: analytic at all pts on C_1 and on C_2, as well as pts in between C_1 and C_2

  $\Updownarrow$ (bad pts of f are outside C_2 or inside C_1)

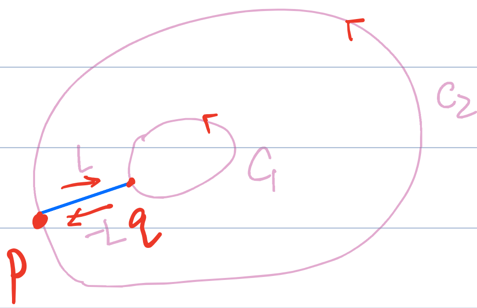


$$\Rightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

x bad pts

Idea: do a surgery.

pf. (NOT required)



Let ① ② ③ ④

$\gamma = C_2 + L - C_1 - L$ ← closed contour

By C.-G. Thm,

$$\int_\gamma f(z)\,dz = 0 \implies$$

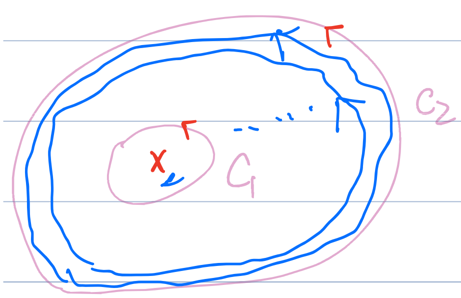
$$0 = \int_{C_2} f(z)\,dz + \int_L f(z)\,dz - \int_{C_1} f(z)\,dz - \int_L f(z)\,dz$$

$$\implies \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Remark: we can memorize the theorem as follows:

Intuitively, if you can deform continuously C_2 to C_1 without touching the "bad pts" of f, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$



x bad pts