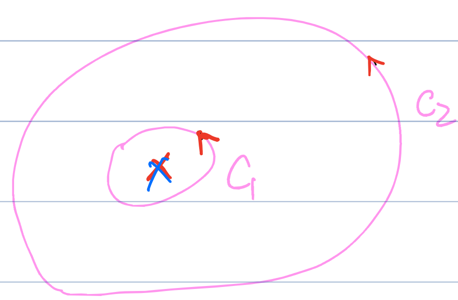
Recall:

In general, we have:

Thm: (principle of deformation of paths)
Keyhole surgery

Let

- C_1, C_2: positively oriented, simply closed
  C_1 is inside C_2
- f: analytic at all pts on C_1 and on C_2, as well as pts in between C_1 and C_2
  $\Updownarrow$ (bad pts of f are outside C_2 or inside C_1)

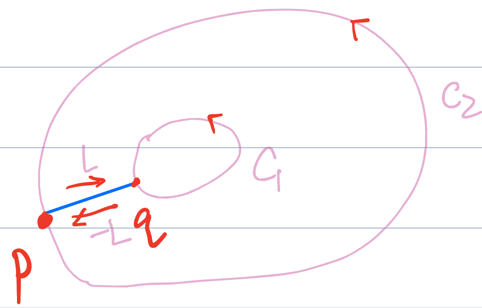


$$\Rightarrow \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

$\times$ bad pts

Idea: do a surgery.

Pf. (NOT required)



Let ① ② ③ ④

$\gamma = C_2 + L - C_1 - L$ ← closed contour

By C.-G. Thm,

$$\int_\gamma f(z)\,dz = 0 \implies$$

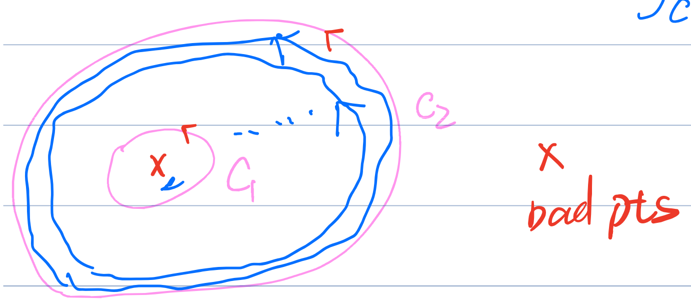
$$0 = \int_{C_2} f(z)\,dz + \int_L f(z)\,dz - \int_{C_1} f(z)\,dz - \int_L f(z)\,dz$$

$$\implies \int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

Remark: we can memorize the theorem as follows:

Intuitively, if you can deform continuously C_2 to C_1 without touching the "bad pts" of f, then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$



Cauchy Integral formula

Thm (Cauchy Integral formula).

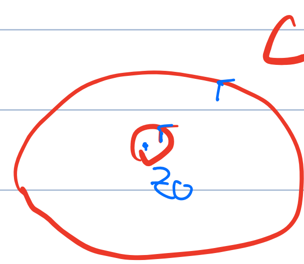
Let C: simply closed contour, positively oriented

$z_0 \in \mathbb{C}$ be a point interior to C

f: analytic at all points inside and on C

$$\Rightarrow \quad f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z_0}\, dw$$

Pf. (NOT required)



We first prove for the case C is a small circle

Lemma. Let C_ρ: $w(t) = z_0 + \rho e^{it}$, $0 \le t \le 2\pi$ $(\rho > 0)$

$$\Rightarrow \quad \int_{C_\rho} \frac{dw}{w - z_0} = 2\pi i$$

Pf of Lemma:

Since $w(t) = z_0 + \rho e^{it}$ $\quad 0 \le t \le 2\pi$

$$\Rightarrow w'(t) = i\rho e^{it}$$

$$\Rightarrow \int_{C_\rho} \frac{dw}{w - z_0} = \int_0^{2\pi} \frac{w'(t)\,dt}{w(t) - z_0}$$

$$= \int_0^{2\pi} \frac{i\cancel{\rho e^{it}}\,dt}{\cancel{\rho e^{it}}}$$

$$= \int_0^{2\pi} i\,dt = 2\pi i$$

Remark: When $z_0 = 0$, the lemma becomes

$$\int_{C_\rho} \frac{dw}{w} = 2\pi i \quad \left(\text{or} \int_{C_\rho} \frac{dz}{z} = 2\pi i \right)$$

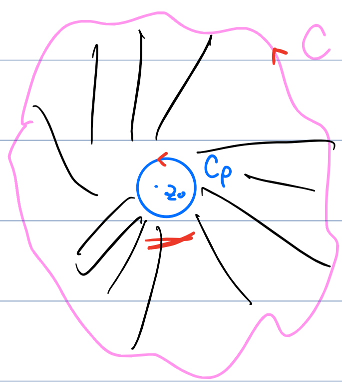
C_ρ: $w(t) = \rho e^{it}$, $0 \le t \le 2\pi$

$$\int_C \frac{f(w)}{w-z_0}\,dw \quad \text{v.s} \quad \int_{C_\rho} \frac{f(w)}{w-z_0}\,dw$$

We continue to prove the Thm.

By the "principle of deformation of paths"

keyhole surgery



$$\Rightarrow \int_C g(w)\,dw = \int_{C_\rho} g(w)\,dw$$

Consider $g(w) = \dfrac{f(w)}{w-z_0}$

Note: bad pts of g inside C:

$$w = z_0$$

Q: when $\rho \to 0+$, $\displaystyle\int_{C_\rho} \frac{f(w)}{w-z_0}\,dw \to \ ?$

A: when $\rho \to 0+$, every $w \in C_\rho$ is close to z_0

(f is analytic $\Rightarrow f$ is continuous)

$\Rightarrow f(w)$ close to $f(z_0)$)

$$\Rightarrow \int_{C_\rho} \frac{f(w)}{w-z_0}\,dw \longrightarrow \int_{C_\rho} \frac{f(z_0)}{w-z_0}\,d$$

But $\displaystyle\int_{C_\rho} \frac{f(z_0)}{w-z_0}\,dw = f(z_0) \int_{C_\rho} \frac{dw}{w-z_0}$

$$= f(z_0) \cdot 2\pi i$$

$$\Rightarrow \int_{C_\rho} \frac{f(w)}{w-z_0}\,dw \to f(z_0) \cdot 2\pi i, \quad \rho \to 0^+$$

Finally since

$$\int_C \frac{f(w)\,dw}{w-z_0} = \int_{C_\rho} \frac{f(w)}{w-z_0}\,dw$$

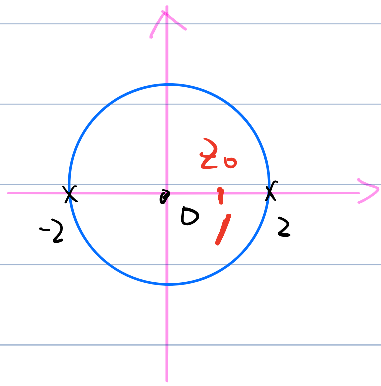
for any small $\rho > 0$

Letting $\rho \to 0+$ ~~at~~ on both sides

$$\Rightarrow \int_C \frac{f(w)}{w-z_0}\,dw = 2\pi i \cdot f(z_0)$$

E.g Let C: $\{|z|=2\}$, positively oriented

Evaluate $\displaystyle\int_C \frac{e^z}{z-1}\,dz$



(Hint: To use C.I.F we need to decide what are f, z_0)

A: Take $f = e^z$ and $z_0 = 1$

Note: $\begin{cases} z_0 \text{ is inside } C \\ f \text{ is entire} \end{cases}$

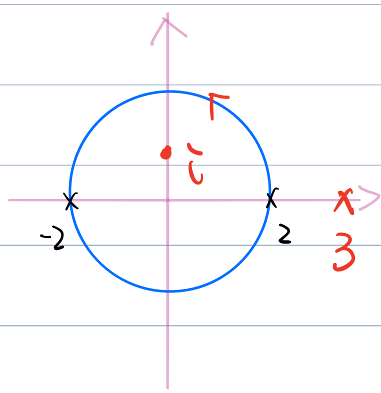
By C.I.F,

$$\int_C \frac{e^z}{z-1}\,dz = \int_C \frac{f(z)}{z-z_0}\,dz$$

$$\overset{\text{C.I.F}}{=} 2\pi i\, f(1) = 2\pi i \cdot e = (2\pi e)\cdot i$$

E.g: let C: $\{|z|=2\}$, positively oriented

Evaluate $\displaystyle\int_C \frac{z^2}{(z-3)^2(z-i)}\,dz$



Warning: In this E.g, we cannot just take

$$f = z^2$$

for otherwise, we cannot pick z_0 s.t

$$z - z_0 = (z-3)^2(z-i)$$

Hint: Consider $\dfrac{z^2}{(z-3)^2(z-i)}$

bad pts: $z=3$, $z=i$

$\Rightarrow$ Take $z_0 = i$

(hence $z - z_0 = z - i$)

A: Take $\begin{cases} z_0 = i \\ f = \dfrac{z^2}{(z-3)^2} \end{cases}$

Note: $\begin{cases} z_0: \text{ inside } C \\ f: \text{ analytic inside } C \text{ and on } C\ ? \end{cases}$

(Yes, the only bad pt of f is $z=3$)

$\begin{cases} z_0 = i \\ f = \dfrac{z^2}{(z-3)^2} \end{cases}$

By C.I.F,

$$\int_C \frac{z^2}{(z-3)^2(z-i)}\,dz = \int_C \frac{f(z)}{z-i}\,dz$$

$$\overset{C.I.F}{=\!=} 2\pi i\, f(i) = 2\pi i \frac{i^2}{(i-3)^2} = \frac{\pi i}{3i-4}$$

Cauchy Integral formula for derivatives

Thm (C.I. formula for derivatives)

Let

C: simply closed, + oriented

z_0: a point inside C

f: analytic inside C and on C

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \boxed{\int_C \frac{f(w)}{(w-z_0)^{n+1}}\, dw}$$

nth derivative of f at z_0

here $n = 1, 2, 3, \ldots$

Pf. (NOT required)

Recall: Cauchy Integral formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z}\, dw \qquad (*)$$

z: any pt inside C

Apply $\frac{d}{dz}$ to $(*)$:

$$\frac{d}{dz} f(z) = \frac{1}{2\pi i} \frac{d}{dz}\left(\int_C \frac{f(w)}{w-z}\, dw \right)$$

by real analysis $= \frac{1}{2\pi i} \int_C \left(\frac{d}{dz} \frac{f(w)}{w-z} \right) dw$

$$= \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)^2}\, dw$$

Apply $\frac{d^2}{dz^2}$ to $(*)$ $\Longrightarrow$

$$\frac{d^2}{dz^2} f(z) = \frac{1}{2\pi i} \frac{d^2}{dz^2} \left(\int_C \frac{f(w)}{w-z}\, dw \right)$$

by real analysis $= \frac{1}{2\pi i} \int_C \frac{d^2}{dz^2} \frac{f(w)}{w-z}\, dw$

$$= \frac{1}{2\pi i} \int_C \frac{2 f(w)}{(w-z)^3}\, dw$$

In general, apply $\frac{d^n}{dz^n}$ to $(*)$ $\Longrightarrow$

$$\frac{d^n}{dz^n} f(z) = \frac{1}{2\pi i} \frac{d^n}{dz^n} \left(\int_C \frac{f(w)}{w-z}\, dw \right)$$

real analysis $= \frac{1}{2\pi i} \int_C \frac{d^n}{dz^n} \left(\frac{f(w)}{w-z} \right) dw$

$$= \frac{1}{2\pi i} \int_C f(w) \frac{d^n}{dz^n} \left(\frac{1}{w-z} \right) dw$$

$$= \frac{1}{2\pi i} \int_C f(w) \frac{n!}{(w-z)^{n+1}}\, dw$$

$$= \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}}\, dw$$

E.x

$$\frac{d^n}{dz^n} \left(\frac{1}{w-z} \right) = \frac{n!}{(w-z)^{n+1}}$$

Remark:

recall C.I.F for derivative

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}}\, dw$$

with $n = 0, 1, 2, 3, \ldots$

Note: it also holds for $n=0$.

why? check $n=0$: $(0! = 1)$

$$LHS = f^{(0)}(z) = f(z)$$

$$RHS = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z)}\, dw$$

$$\Rightarrow \quad f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w-z}\, dw$$

(when $n=0$, it becomes C.I.F)