Reminder:

Final Exam:

3:00 - 5:30pm, Monday 03/18

0121 RWAC

It will cover page 62 - 170

(section 21 - 57).

Review class: on Friday.

Recall:

Cauchy Integral formula for derivatives

Thm (C.I. formula for derivatives)

Let

C: simply closed, + oriented

z_0: a point inside C

f: analytic inside C and on C

$\Rightarrow$ (1) f is infinitely complex diff'ble

(2) Moreover,

we have $f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z_0)^{n+1}} \, dw$

where $n = 0, 1, 2, 3, \ldots$

Remark: when $n=0$, it becomes

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(w)}{(w-z_0)} \, dw$$

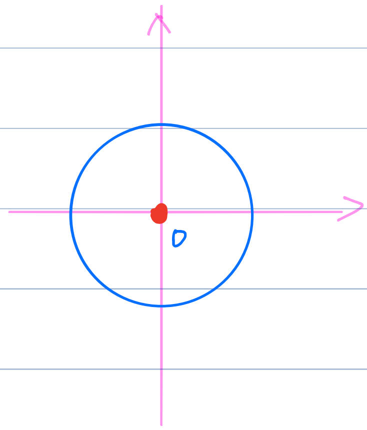
C. I. F

Tips: we often use the formula as

$$\int_C \frac{f(w)}{(w-z_0)^{n+1}}\,dw = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

E.g. $\int_C \frac{e^{3z}}{z^2}\, dz$

where $C: \{|z|=1\}$ positively oriented



Q: what are the bad pts of g?

A: $z=0$.

A: Take $z_0=0$, $n=1$,

$f(z)=e^{3z}$

Note: · 0 is inside C

· f: analytic inside C and on C

Then

$$\int_C \frac{e^{3z}}{z^2}\, dz = \int_C \frac{f(z)\, dz}{(z-0)^2}$$

$$\underset{\text{for derivative}}{\overset{\text{C.I.F}}{=}} \frac{2\pi i}{1!} f'(0)$$

Recall, $f = e^{3z}$

$$f' = 3e^{3z}$$

$$f'(0) = 3 \cdot e^{0} = 3$$

Hence

$$\int_C \frac{e^{3z}}{z^2}\,dz = 2\pi i \cdot 3$$

$$= 6\pi i$$

E.g. Compute $\displaystyle\int_C \frac{dz}{(z-z_0)^{n+1}}$, $n \in \mathbb{Z}$, $n \geq 0$

where C: simply closed contour, positively oriented that encloses z_0

A: Take $f = 1$,

Note: f: analytic inside C and on C

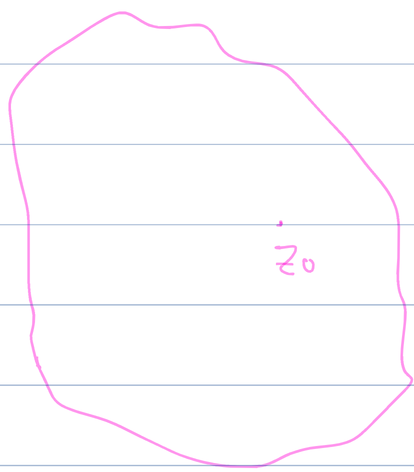
$\Rightarrow$ By C.I.F. for D.

$$\int_C \frac{dz}{(z-z_0)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

$$= \begin{cases} \dfrac{2\pi i}{0!} f(z_0) = 2\pi i & n = 0 \\ 0 & n \geq 1 \end{cases}$$

E.g. Evaluate $\int_C \frac{z^2}{(z-z_0)^{n+1}}\,dz$, $n \in \mathbb{Z}$, $n \geq 0$.

where C: simply closed contour, positively oriented that encloses z_0



A: Take $f = z^2$

By C.I.F for D.

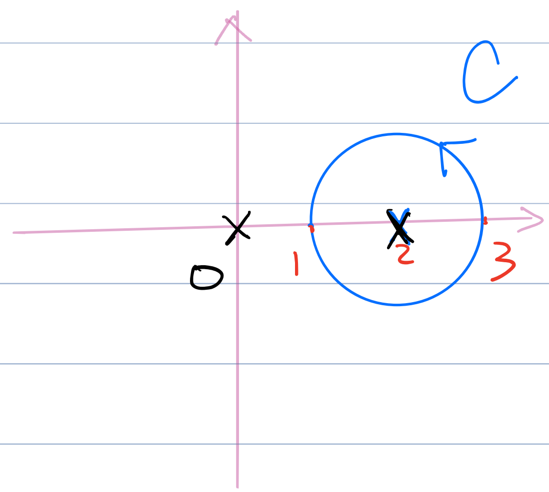
Hint: $0! = 1$

$$\Rightarrow \int_C \frac{z^2}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

$$= \begin{cases} 2\pi i\, z_0^2, & n=0; \\ \frac{2\pi i}{1!} 2z_0 = 4\pi i z_0, & n=1; \\ \frac{2\pi i}{2!} \cdot 2 = 2\pi i & n=2 \\ 0 & n \geq 3 \end{cases}$$

E.g Evaluate $\int_C \frac{1}{z^2(z-2)^2}\,dz$ (g)

where $C = \{|z-2| = 1\}$, "+" oriented



Q: What are the bad pts of g? which ones are inside C?

A: $z=0$ and $z=2$

only $z=2$ is inside C.

$f = \frac{1}{z^2}$

only bad at $z=0$

A: Take $z_0 = 2$,

(Hint: choose f, n s.t

$$\int_C \frac{1}{z^2(z-2)^2}\,dz = \int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$)

Choose. $n=1$, $f=\frac{1}{z^2}$

Note: • z_0 is inside C

• f is analytic $\begin{cases} \text{inside } C \\ \text{on } C \end{cases}$

$\Rightarrow$

$$\int_C \frac{1}{z^2(z-2)^2}\,dz = \int_C \frac{f(z)}{(z-2)^2}\,dz$$

$$\underline{\underline{\text{C.I.F for D.}}}\ \frac{2\pi i}{1!} f'(2)$$

$$= 2\pi i\,(-2)\,2^{-3}$$

$$= -\frac{1}{2}\pi i$$

$$\boxed{\begin{aligned} f &= \frac{1}{z^2} \\ &= z^{-2} \\ \Rightarrow\ f' &= -2z^{-3} \end{aligned}}$$

Remark: Indeed,

Cauchy Integral formula (C. I. F)

$\Rightarrow$ Cauchy - Goursat Thm

why?

Recall: Cauchy-Goursat

If • C: simply closed

• f: analytic inside C and on C

$$\Rightarrow \int_C f(z)\,dz = 0$$

C Pf of C.-G. Thm:

Pick any z_0 inside C.

Let $h(z) = f(z)(z - z_0)$

$$\Rightarrow \int_C f(z)\,dz = \int_C \frac{h(z)}{z - z_0}\,dz$$

$$= \int_C \frac{h(z)}{(z - z_0)^{0+1}}\,dz$$

$$= \frac{2\pi i}{0!} h^{(0)}(z_0)$$

$$= 2\pi i \, h(z_0)$$

$$= 0$$

Note:

$h(z) = f(z)(z - z_0)$

$\Rightarrow$

$h(z_0) = 0$

Summary:

Q: How many methods do we have for computing contour integrals?

A:

① (By definition) If $C: z(t),\ a \le t \le b$

$$\Rightarrow \int_C f(z)\,dz = \int_a^b f(z(t))\cdot z'(t)\,dt$$

② (Finding antiderivative)

When you compute $\int_C f(z)\,dz$, if you can find F, D s.t

$$\begin{cases} C: \text{is entirely in } D \\ F: \text{analytic in } D \\ \quad F' = f \text{ on } D \end{cases}$$

$$\Rightarrow \int_C f(z)\,dz = F(z(b)) - F(z(a))$$

If in addition, C is closed $(z(a) = z(b))$

$$\Rightarrow \int_C f(z)\,dz = 0$$

③ (Cauchy - Goursat Thm)

If C: simply closed

f: analytic inside and on C

$$\Rightarrow \int_C f(z)\,dz = 0$$

④ (C.I.F for derivatives)

Thm: If

- C: simply closed. "+" oriented

  f: analytic inside and on C

  z_0: a point inside C

$$\Rightarrow \int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

require c to be closed

Method ①, ②, ③, ④

Q: When to use which method?

A: Some tips:

(a) When C is not closed, choose from ①, ②

- If you cannot parametrize C, do NOT use ①
- If you cannot find the anti-derivative of f, do NOT use ②;

(b) When C is simply closed ("+" oriented), and f is a quotient of two analytic functions: $f = \frac{p}{q}$ (very often p, q are polynomials), use ③ or ④

③ v.s. ④ $\int_C g(z)\,dz$

Tips: check how many bad pts of g are there inside C? 3 Scenarios

(I)	(II)	(III)
No bad pts inside C	1 bad pt z_0 inside C	≥ 2 bad pts inside C
↓	↓	↓
use ③	use ④	Next quarter

Q: What if there is a bad pt right on the contour C?

A: No worries! We will never give such a problem with bad pts on C.