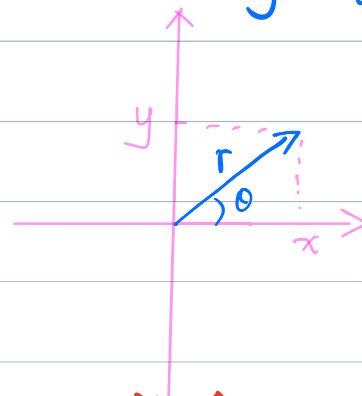
Recall: 2 ways to express a complex number $z \in \mathbb{C}$

$x+iy \leftrightarrow (x,y)$

① rectangular form:

$$z = x + iy$$



② polar form:

$$z = re^{i\theta}$$

$e^{i\theta} \triangleq \cos\theta + i\sin\theta$

Here • $r = \sqrt{x^2+y^2}$: modulus of z

Notation: $|z| = r$

• argument

$\arg z$ = all possible choices of θ

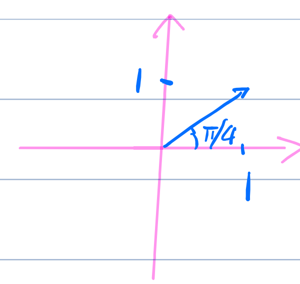
$\operatorname{Arg} z$ = the special choice of θ within $(-\pi, \pi]$

E.g:

$\arg(1+i) = \frac{\pi}{4} + 2k\pi, \ k \in \mathbb{Z}$

$\operatorname{Arg}(1+i) = \frac{\pi}{4}$

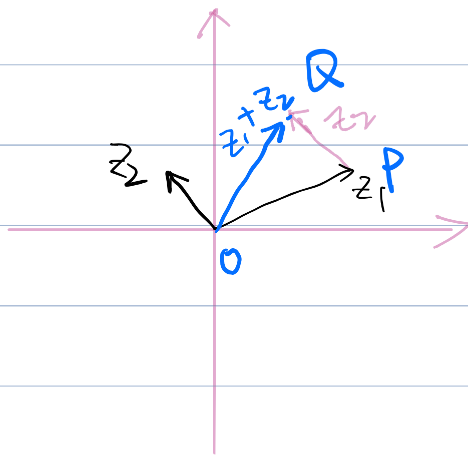
principal argument



Triangle inequality:

$$|z_1+z_2| \le |z_1|+|z_2| \Longleftrightarrow |\overrightarrow{OQ}| \le |\overrightarrow{OP}|+|\overrightarrow{PQ}|$$

Geometric meaning: the shortest path between two points is a straight line segment.



$$|z_1+z_2| = |\overrightarrow{OQ}|$$
$$|z_1| = |\overrightarrow{OP}|$$
$$|z_2| = |\overrightarrow{PQ}|$$

we know $|z_1+z_2| \le |z_1|+|z_2|$.

Q: when $|z_1+z_2| = |z_1|+|z_2|$?

A: They are equal when z_1, z_2 point to the same direction

That is, to make $|z_1+z_2|$ as large as possible. we should make z_1, z_2 point to the same direction.

E.x: How to make $|z_1+z_2|$ as small as possible?

Useful formulas:

① $|e^{i\theta}|=1$

why? recall if $z=x+iy \Rightarrow |z|=\sqrt{x^2+y^2}$

recall $e^{i\theta}=\cos\theta+i\sin\theta \Rightarrow |e^{i\theta}|$

$=\sqrt{\cos^2\theta+\sin^2\theta}$

$=1$

② $z\bar{z}=|z|^2$

why? write $z=x+iy \Rightarrow \bar{z}=x-iy$

$z\cdot\bar{z}=(x+iy)(x-iy)$

$=x^2+y^2=|z|^2$

③ $e^{i0} = \cos 0 + i\sin 0 = 1 + i0 = 1$

$e^{i\pi} = \cos\pi + i\sin\pi = -1 + i0 = -1$

$(\text{or } e^{i\pi} + 1 = 0)$

Why do we introduce the polar form

$z = e^{i\theta}$?

A: multiple complex numbers in polar form is easier.

Important formula:

$$\boxed{\begin{array}{l}\text{If } z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2}, \\ \text{then} \qquad\qquad\qquad\qquad\qquad (*) \\ \qquad z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}\end{array}}$$

Pf: will do it later.

We first discuss some consequences of (*)

① $|z_1 z_2| = \left| r_1 r_2 e^{i(\theta_1+\theta_2)} \right|$

$= r_1 r_2$

Recall:
$|e^{i\theta}| = 1$

② If $z = re^{i\theta}$,
then $z^{-1} = \frac{1}{z} = \frac{1}{r} e^{i(-\theta)}$

why? verify $z \cdot z^{-1} = 1$:

$$z \cdot z^{-1} = re^{i\theta} \frac{1}{r} e^{i(-\theta)}$$

$$= (r \cdot \frac{1}{r}) e^{i(\theta-\theta)}$$

$$= 1 \cdot e^{i0} = 1$$

③ $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$, where $z_2 \neq 0$

Pf: Ex. Hint: use ②

④ If $z = re^{i\theta}$, then

$$z^2 = z \cdot z = re^{i\theta} \cdot re^{i\theta}$$
$$= r^2 e^{i(2\theta)}$$

$$z^3 = z^2 \cdot z$$
$$= r^2 e^{i(2\theta)} \cdot re^{i\theta}$$
$$= r^3 e^{i(3\theta)}$$

In general,

$$\boxed{\text{If } z = re^{i\theta} \Rightarrow z^n = r^n e^{i(n\theta)}, \quad n \geq 0,\ n \in \mathbb{Z}}$$

If $n = 0$, it means

$$z^0 = r^0 e^{i0} = 1$$

In the special case $r=1$, $z = re^{i\theta} = e^{i\theta}$

then the above formula $\Rightarrow$

$$(e^{i\theta})^n = e^{in\theta}$$

De Moirre's formula

Now we prove the important formula:

$$\boxed{\begin{array}{l} \text{If } z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2}, \\ \text{then} \qquad\qquad\qquad\qquad (*) \\ \qquad z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} \end{array}}$$

Good news: proof NOT required! ◡̈

Pf: Since $z_1 = r_1 e^{i\theta_1}$,

$\Rightarrow z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

Since $z_2 = r_2 e^{i\theta_2}$,

$\Rightarrow z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then

$$z_1 z_2 = r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2 \left[(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2) \right]$$

$$= r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right]$$

Recall Trig function:

$$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$

$$\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2.$$

$$= r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2) \right]$$

$$= r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

Q: T or F

" For all $z_1, z_2 \neq 0$,

(1) $\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2$ "

A: This statement is False.

why?

- For some choices of z_1, z_2, (1) holds.

E.g. Take $z_1 = 1$, $z_2 = i \Rightarrow z_1 z_2 = i$

$$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(i) = \frac{\pi}{2}$$

$\mathrm{Arg}(z_1) = \mathrm{Arg}(1) = 0$, $\mathrm{Arg}(z_2) = \mathrm{Arg}(i) = \frac{\pi}{2}$

- Hower, for some other choices of z_1, z_2, (1) fails.

Eg: Take $z_1 = i$, $z_2 = -1$

$$\Rightarrow z_1 z_2 = -i$$

$$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(-i) = -\frac{\pi}{2}$$

$$\operatorname{Arg}(z_1) = \operatorname{Arg}(i) = \frac{\pi}{2}$$

$$\operatorname{Arg}(z_2) = \operatorname{Arg}(-1) = \pi$$

$$LHS = \operatorname{Arg}(z_1 z_2) = -\frac{\pi}{2}$$

$$RHS = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) = \frac{\pi}{2} + \pi = \frac{3\pi}{2}$$

Another application of polar form:

Next topic: How to find nth root?

Q: Given $z_0 \in \mathbb{C}$, and $n > 0$, $n \in \mathbb{Z}$

solve $z^n = z_0$

Note: when $z_0 = 0$, $z^n = 0$ has only one solution $z = 0$

we will thus next assume $z_0 \neq 0$.

Warm-up problem: say $z_0 = 1+i$, $n = 2$

E.g: solve $z^2 = 1+i$

Step 1: write RHS in polar form:

$z = re^{i\theta}$, where $\theta = \operatorname{Arg} z_0$

$$\text{RHS} = 1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$$

Step 2: write $z = re^{i\theta}$

(r, θ to be decided)