Another application of polar form:

Next topic: How to find nth root?

Q: Given $z_0 \in \mathbb{C}$, and $n > 0$, $n \in \mathbb{Z}$

solve $z^n = z_0$

Note: when $z_0 = 0$, $z^n = 0$ has only one solution $z = 0$

We will thus next assume $z_0 \neq 0$.

Warm-up problem: say $z_0 = 1+i$, $n=2$

E.g.: solve $z^2 = 1+i$

Step 1: write RHS in polar form:

$z_0 = re^{i\theta}$, where $\theta = \mathrm{Arg}\, z_0$

$$\mathrm{RHS} = 1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$$

Step 2: write $z = re^{i\theta}$, $r > 0$

(r, θ to be decided)

and substitute into the equation:

$$(re^{i\theta})^2 = 1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$$

$$\Rightarrow \quad r^2 e^{i2\theta} = \sqrt{2}e^{i\frac{\pi}{4}}$$

Q: How to make two polar forms equal?

A: $r_1 e^{i\theta_1} = r_2 e^{i\theta_2}$, $r_1, r_2 > 0$

$$\Leftrightarrow \begin{cases} r_1 = r_2 \\ \theta_1 = \theta_2 + 2k\pi, \ k \in \mathbb{Z} \end{cases}$$

Then we must have

$$\begin{cases} r^2 = \sqrt{2} = 2^{\frac{1}{2}} & ① \\ 2\theta = \frac{\pi}{4} + 2k\pi, \ k \in \mathbb{Z} & ② \end{cases}$$

That is, $\begin{cases} r = 2^{\frac{1}{4}} \\ \theta = \frac{\pi}{8} + k\pi, \ k \in \mathbb{Z} \end{cases}$ and $z = re^{i\theta}$

Note:

- If $k=0$, $\Rightarrow z = re^{i\theta} = 2^{\frac{1}{4}} e^{i(\frac{\pi}{8})}$
- If $k=1$, $\Rightarrow z = re^{i\theta} = 2^{\frac{1}{4}} e^{i(\frac{\pi}{8}+\pi)}$
- If $k=2 \Rightarrow z = re^{i\theta} = 2^{\frac{1}{4}} e^{i(\frac{\pi}{8}+2\pi)}$
- If $k=3 \Rightarrow z = 2^{\frac{1}{4}} e^{i(\frac{\pi}{8}+3\pi)}$

Similarly,

- If $k=-1$, $\Rightarrow z = 2^{\frac{1}{4}} e^{i(\frac{\pi}{8}-\pi)}$
$= 2^{\frac{1}{4}} e^{i(\frac{\pi}{8}+\pi)}$

- If $k=-2$, $\Rightarrow z = 2^{\frac{1}{4}} e^{i(\frac{\pi}{8}-2\pi)}$
$= 2^{\frac{1}{4}} e^{i\frac{\pi}{8}}$

$\vdots$

Hence instead of writing the final answer as

" $z = 2^{\frac{1}{4}} e^{i(\frac{\pi}{8} + k\pi)}$, $k \in \mathbb{Z}$ (or $k = 0, \pm 1, \pm 2, \cdots$) "

It's much better to write it as

$$\boxed{z = 2^{\frac{1}{4}} e^{i(\frac{\pi}{8} + k\pi)}, \quad k = 0, 1}$$

Summarize:

Q: solve $z^2 = 1 + i$

A: $z = 2^{\frac{1}{4}} e^{i(\frac{\pi}{8} + k\pi)}$, $k = 0, 1$

Next: a slightly more complicated Q:

Q: $z^3 = 1 + i = \sqrt{2} e^{i\frac{\pi}{4}}$

A: write $z = re^{i\theta}$, $r > 0$ $\Rightarrow$

$z^3 = 1 + i$

$\Rightarrow (re^{i\theta})^3 = r^3 e^{3i\theta} = \sqrt{2} e^{i\frac{\pi}{4}}$

$$\Rightarrow \begin{cases} r^3 = \sqrt{2} = 2^{\frac{1}{2}} & \Rightarrow \quad r = 2^{\frac{1}{6}} \\ 3\theta = \frac{\pi}{4} + 2k\pi, \ k \in \mathbb{Z} & \Rightarrow \quad \theta = \frac{\pi}{12} + \frac{2k\pi}{3}, \quad k \in \mathbb{Z} \end{cases}$$

$$\Rightarrow \quad z = \underbrace{2^{\frac{1}{6}} e^{i(\frac{\pi}{12} + \frac{2k}{3}\pi)}}, \quad k \in \mathbb{Z}$$

- If $k=0$, $z = 2^{\frac{1}{6}} e^{i\frac{\pi}{12}}$
- If $k=1$, $z = \cdots$
- If $k=2$, $z = \cdots$
- If $k=3$, $z = 2^{\frac{1}{6}} e^{i(\frac{\pi}{12} + \frac{2\cdot 3\pi}{3})} = 2^{\frac{1}{6}} e^{i(\frac{\pi}{12} + 2\pi)}$
- If $k=4$, $z = \cdots$

$\vdots$

Conclusion: The above are repeating $k = 0, 1, 2, \cdots$.

Then we should write the answer as

$$\boxed{z = 2^{\frac{1}{6}} e^{i(\frac{\pi}{12} + \frac{2}{3}k\pi)}, \quad k = 0, 1, 2,}$$

Summarize:

Q1: $z^2 = 1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$

A1: $z = 2^{\frac{1}{4}} e^{i(\frac{\pi}{8}+k\pi)}$, $k=0,1$.

Q2: $z^3 = 1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$

A2: $z = 2^{\frac{1}{6}} e^{i(\frac{\pi}{12}+\frac{2}{3}k\pi)}$, $k=0,1,2$

Guess what is the answer to:

Q: $z^n = 1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$, where $n>0$, $n\in\mathbb{Z}$

A: $z = \cdots\cdots$, $\underbrace{k=0,1,\cdots,n-1}_{n \text{ choices}}$

Now we consider the very general Q:

Q: Given $z_0 \neq 0$, $n>0$, $n\in\mathbb{Z}$, solve

$$z^n = z_0$$

Algorithm:

Step 1: write z_0 in the polar form

$z_0 = r_0 e^{i\theta_0}$ with $\theta_0 = \operatorname{Arg} z_0$

Step2: Substitute $z = re^{i\theta}, r>0,$ into the eqn.

$$\Rightarrow \quad r^n e^{in\theta} = r_0 e^{i\theta_0}$$

$$\Rightarrow \begin{cases} r^n = r_0 \\ n\theta = \theta_0 + 2k\pi, \ k \in \mathbb{Z} \end{cases}$$

$$\Rightarrow \begin{cases} r = r_0^{\frac{1}{n}} \\ \theta = \frac{\theta_0}{n} + \frac{2k}{n}\pi, \ k \in \mathbb{Z} \end{cases}$$

$$\Rightarrow$$

$$\boxed{\begin{aligned} z &= re^{i\theta} \\ &= r_0^{\frac{1}{n}} e^{i\left(\frac{\theta_0}{n} + \frac{2k}{n}\pi\right)}, \ k = 0, 1, \cdots, n-1 \end{aligned}}$$

Remark: when $k=0$, $\Rightarrow$

$$z = r_0^{\frac{1}{n}} e^{i\frac{\theta_0}{n}}$$

It is called the principal nth root of z_0

A special case of interest: $z_0 = 1$.

(that is to solve $z^n = 1$)

Q: find all nth roots of unity:

solve $z^n = 1$.

A: Step 1: $1 = 1 e^{i \cdot 0}$

Step 2: $z = re^{i\theta}$, $\underbrace{r > 0}$,

Then "$z^n = 1$" $\Rightarrow$

$$r^n e^{in\theta} = 1 \cdot e^{i0}$$

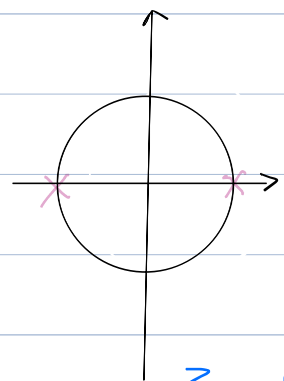
$$\Rightarrow \begin{cases} r^n = 1 \\ n\theta = 0 + 2k\pi, \quad k \in \mathbb{Z} \end{cases}$$

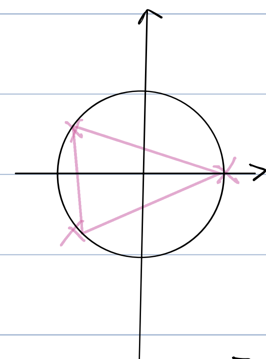
$$\Rightarrow \begin{cases} r = 1^{\frac{1}{n}} = 1 \\ \theta = 0 + \frac{2k\pi}{n} = \frac{2k\pi}{n}, \quad k \in \mathbb{Z} \end{cases}$$

$$\Longrightarrow \boxed{\begin{aligned} z = re^{i\theta} &= e^{i\frac{2k\pi}{n}}, \quad k = 0, 1, \cdots, n-1 \\ &= \left(e^{\frac{i2\pi}{n}}\right)^k \quad k = 0, 1, \cdots, n-1 \end{aligned}}$$

$$\boxed{\text{Let } w_n = e^{i\frac{2\pi}{n}}}$$

Note: The points $\{1 = w_n^0, w_n, w_n^2, \cdots, (w_n)^{n-1}\}$ form the vertices of a regular n-polygon.



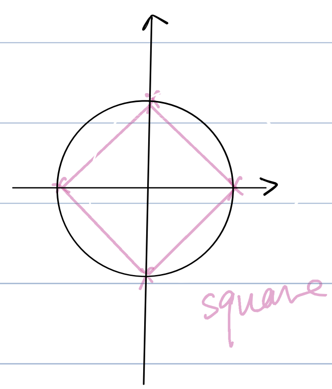


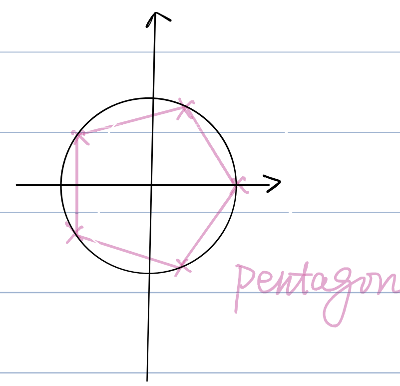
$n=2 \Rightarrow z^2 = 1$

$\Rightarrow z \in \{1, e^{i\pi} = -1\}$

$n=3 \Rightarrow z^3 = 1$

$\Rightarrow z \in \{1, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}\}$





$n=4 \Rightarrow z^4 = 1$

$\Rightarrow z \in \{1, e^{i\frac{2\pi}{4}}, e^{i\frac{4\pi}{4}}, e^{i\frac{6\pi}{4}}\}$

$n=5 \Rightarrow z^5 = 1$

$\Rightarrow z \in \{1, e^{i\frac{2\pi}{5}}, e^{i\frac{4\pi}{5}}, e^{i\frac{6\pi}{5}}, e^{i\frac{8\pi}{5}}\}$

Chapter 2. Region in the complex plane

Q1: What is "the complex plane"?

A1: Recall every complex number

$z = x + iy \in \mathbb{C}$ can be identified

with a vector $(x, y) \in \mathbb{R}^2$ (or xy-plane)

under this point of view,

$\mathbb{C} \simeq \mathbb{R}^2$;

a point on xy-plane represents

a complex number

In this sense,

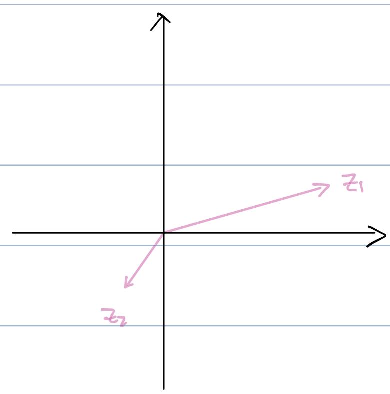
xy-plane is also called the complex plane,

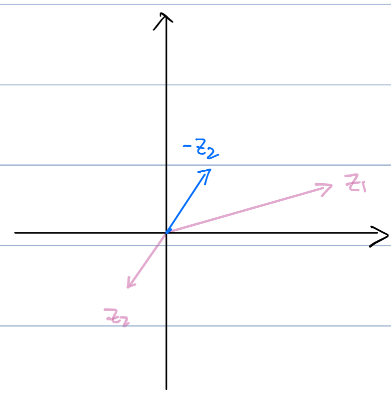
or simply the z-plane.

Q2: Let $z_1, z_2 \in \mathbb{C}$, what is geometric meaning of $|z_1 - z_2|$?

A2: $|z_1 - z_2|$ = distance between z_1 and z_2 on the z-plane

why? check the picture!

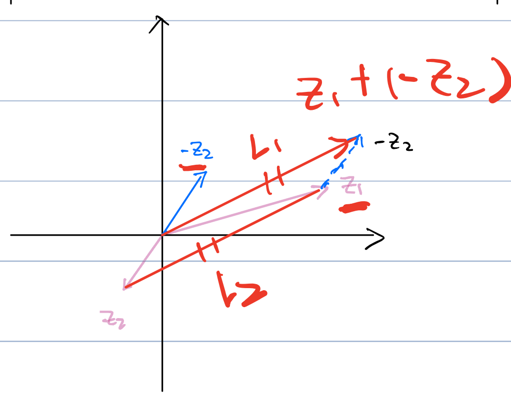




Idea:

$z_1 - z_2$

$= z_1 + (-z_2)$



$L_1 = L_2$

$\downarrow$

the distance between z_1 and z_2

Note:

$z_1 - z_2$

$= z_1 + (-z_2)$

= the vector from z_2 to z_1.

E.g.: Describe (the geometric meaning) of the set

$$\{z \in \mathbb{C} : |z-3| = 2\}$$

A:

Note: $|z-3| = 2 \iff$ distance between z and 3 equals 2

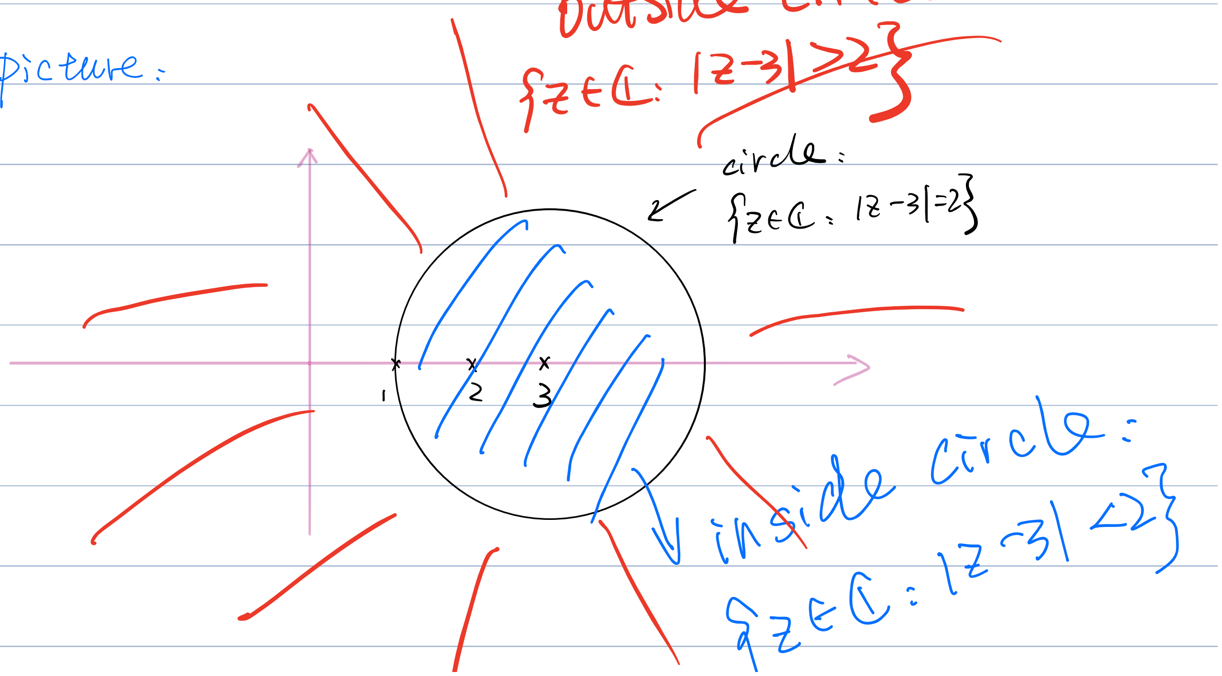
Thus:

$$\{z \in \mathbb{C} : |z-3| = 2\}$$

= the set of points z whose distance to 3 equals 2

= circle of radius 2 centered at 3

Picture:



In the future, we will often use the following shorthands:

$$\{z \in \mathbb{C} : |z-3| = 2\} = \{|z-3| = 2\}$$

$$\{z \in \mathbb{C} : |z| < 1\} = \{|z| < 1\}$$

Some important definitions:

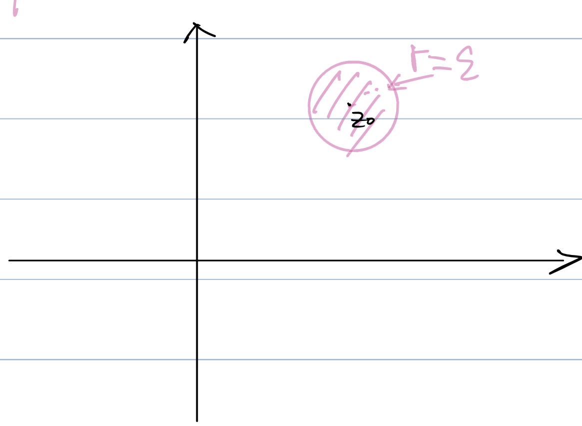
Def^n: Let $z_0 \in \mathbb{C}$, $\varepsilon > 0$, ← often will be a very small number

The set $\{|z - z_0| < \varepsilon\}$

is called a neighborhood of z_0

or a ε-neighborhood of z_0

picture:



Note: It is the inside of the circle

Def^n: Let S be a subset of $\mathbb{C}$
and let $z_0 \in S$

z_0 is called an interior pt of S

$\Longleftrightarrow$ S contains a neighborhood of z_0

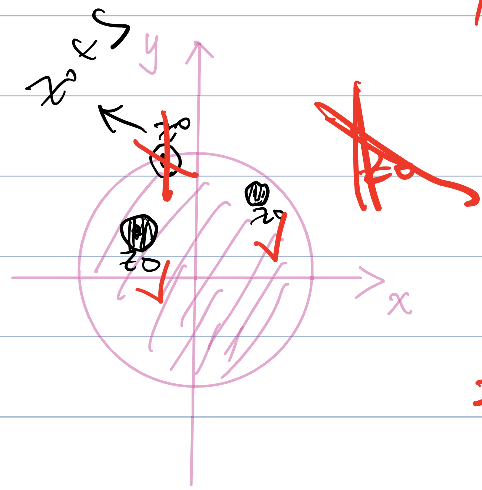
($\Longleftrightarrow$ If we can find some small $\varepsilon > 0$, such that
the small disk $\{|z - z_0| < \varepsilon\} \subseteq S$)

Remark: For z_0 to be an interior pt of S
first of all z_0 must be in S.

E.g: Find all interior pts of the set S

① $S = \{ |z| \leq 1 \}$.

$|z-0| \leq 1 \rightarrow$ contains both inside and the circle itself.



interior pts of $S = \{ z \in \mathbb{C} : |z| < 1 \}$

② $S = \{ \operatorname{Im} z > 0 \}$