Chapter 1. Region in the complex plane

Q1: What is "the complex plane"?

A1: Recall every complex number

$z = x + iy \in \mathbb{C}$ can be identified

with a vector $(x, y) \in \mathbb{R}^2$ (or xy-plane)

under this point of view,

$$\mathbb{C} \simeq \mathbb{R}^2;$$

a point on xy-plane represents

a complex number

In this sense,

xy-plane is also called the complex plane,

or simply the z-plane.

Chapter 1. Region in the complex plane

(some basic $\underbrace{\text{topology}}$ in $\mathbb{C}$)

"What are open. closed sets"

Some important definitions:

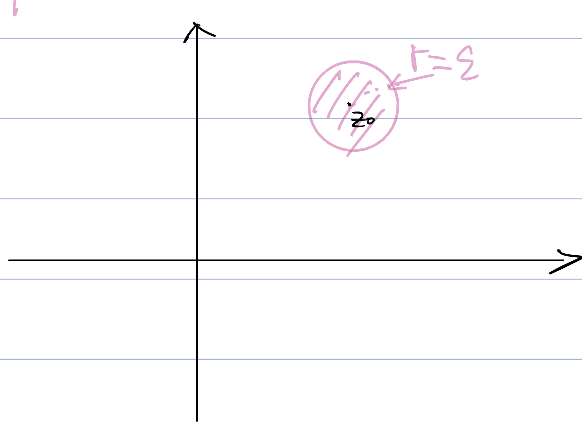
Def^n: Let $z_0 \in \mathbb{C}$, $\varepsilon > 0$, ← often will be a very small number

The set $\{|z - z_0| < \varepsilon\}$

is called a neighborhood of z_0

or an ε-neighborhood of z_0

picture:



Note: It is the inside of the circle

Def^n: Let S be a subset of $\mathbb{C}$

and let $z_0 \in S$

z_0 is called an interior pt of S

$\Longleftrightarrow$ S contains a neighborhood of z_0

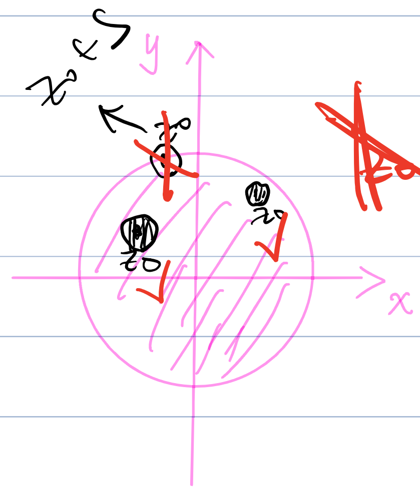
($\Longleftrightarrow$ If we can find some small $\varepsilon > 0$, such that the small disk $\{|z - z_0| < \varepsilon\} \subseteq S$)

Remark: For z_0 to be an interior pt of S first of all z_0 must be in S.

E.g: Find all interior pts of the set S

① $S = \{ |z| \le 1 \}$.

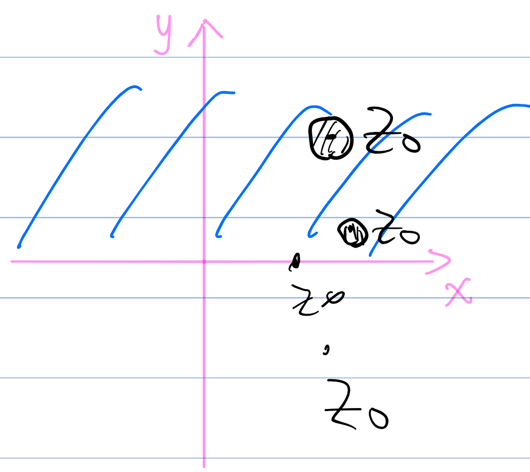
$|z-0| \le 1 \rightarrow$ contains both inside and the circle itself.



interior pts of S

$= \{ z \in \mathbb{C} : |z| < 1 \}$

② $S = \{ \operatorname{Im} z > 0 \} = \{ x+iy : y > 0 \}$

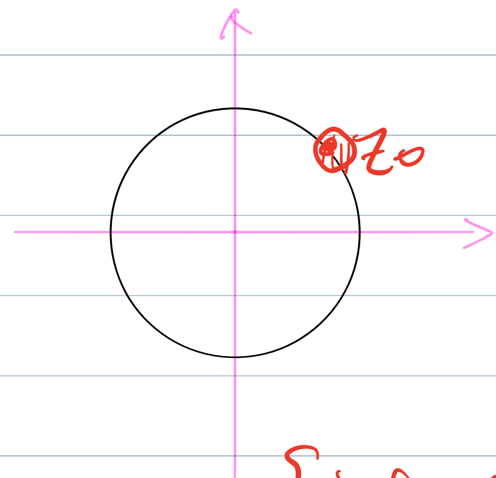


{Interior pts of S}

$= \{ \operatorname{Im} z > 0 \}$

$= S$

③ $S = \{|z| = 1\} \to$ circle



Hence, S has no interior pts;

i.e.,

$\{\text{interior pts of } S\} = \phi$.

Def^n: $S \subseteq \mathbb{C}$ is called open (or an open set)

$\Longleftrightarrow$ every point in S is an interior point of S

(That is,

$\{\text{interior pts of } S\} = S$)

E.g. Is the set S open?

① $S = \{|z| \le 1\}$ E.x

② $S = \{\operatorname{Im} z > 0\}$ open

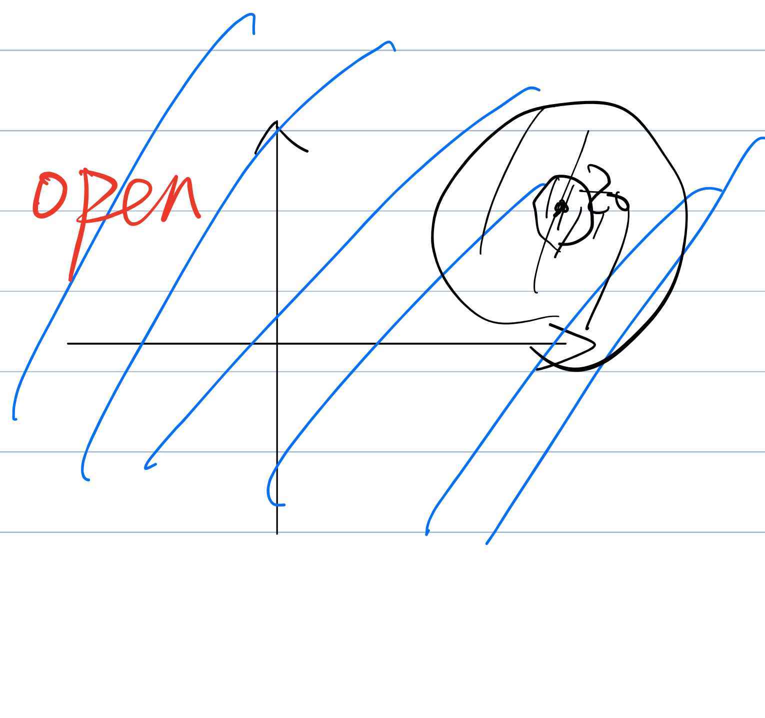
③ $S = \{|z| = 1\}$ NOT open

④ $S = \{|z| < 1\}$ E.x

⑤ $S = \{0 < |z| < 1\}$ E.x

⑥ $S = \mathbb{C}$: open

$\{$interior pts of $S\} = \mathbb{C}$
$= S$



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Remark: The empty set ϕ is also open.

Why? to say "ϕ is open"

it is equivalent to say:

"$\underbrace{\text{If } z \in \phi}_{p}$, $\overset{\text{then}}{\Rightarrow}$ $\underbrace{z \text{ is an interior pt of } \phi}_{q}$"

always false

Boundaries and closed sets

Def^n: Given $S \subseteq \mathbb{C}$, let $\overset{\text{complement of } S}{S^c} = \mathbb{C} - S$

$$= \{ z \in \mathbb{C} : z \notin S \}$$

E.g.: If $S = \{ |z| < 1 \}$

$$S^c = \{ |z| \geq 1 \}$$

Def^n: Let $S \subseteq \mathbb{C}$, $z_0 \notin S$, we say

z_0 is exterior to S

(or z_0 is an exterior point of S)

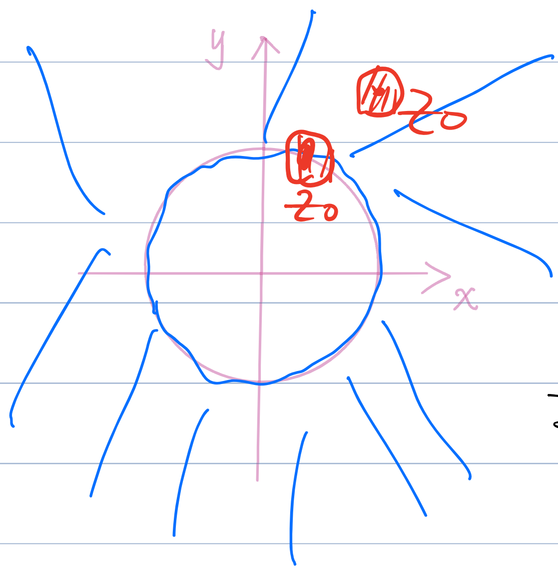
$\Longleftrightarrow$ z_0 is an interior pt of S^c

Remark: For z_0 to be exterior to S, first of all $z_0 \notin S$.

(that is, $z_0 \in S^c$)

E.g: Find S^c and exterior points of S

① $S=\{|z|<1\} \Rightarrow S^c=\{|z|\ge 1\}$.

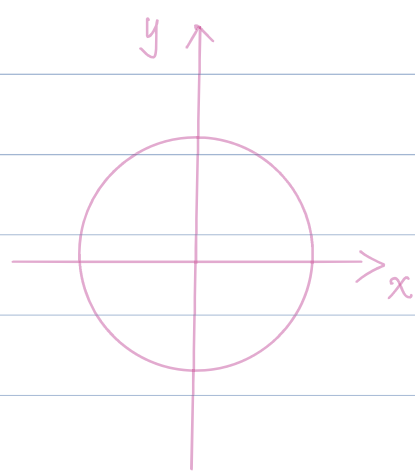


$\{\text{exterior pts of } S\}$

$=\{\text{interior pts of } S^c\}$

$=\{|z|>1\}$.

② $S=\{|z|=1\}$



E.x

③ $S = \{|z| \geq 1\}$.

Ex

Def^n: Given $S \subseteq \mathbb{C}$, $z_0 \in \mathbb{C}$ might be in S, might NOT be in S.

z_0 is called a boundary point of S

$\iff$ z_0 is NOT an interior pt of S

and is NOT an exterior pt of S

Remark: z_0 is a boundary point of S

$\iff$ for all $\varepsilon > 0$, the neighborhood of z_0. $\{|z - z_0| < \varepsilon\}$ contains some pt in S, and some pt in S^c.

Notation: $bS = \{\text{boundary pts of } S\}$

the boundary of S

Remark: Given $S \subseteq \mathbb{C}$, for every $z_0 \in \mathbb{C}$, one and only one of the following must occur:

(1) z_0 is an interior pt of S

(2) z_0 is an exterior pt of S

(3) z_0 is a bdry pt of S

Important formula:

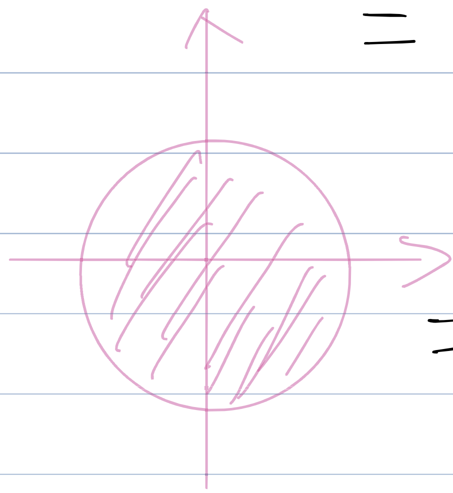
$$bS = \mathbb{C} - \{\text{interior pts of } S\} - \{\text{exterior pts of } S\}$$

E.g: Find the boundary of S

① $S = \{|z| < 1\}$.

Note: $\{\text{interior pts of } S\}$

$= S = \{|z| < 1\}$



$S^c = \{|z| \geq 1\}$

$\{\text{exterior pts of } S\}$

$= \{\text{interior pts of } S^c\}$

$= \{|z| > 1\}$

Hence, $bS = \{|z| = 1\}$.

② $S = \{0 < |z| < 1\}$

$bS = \{0\} \cup \{|z| = 1\}$.

Def^n: $S \subseteq \mathbb{C}$ is closed

Def^n $\longrightarrow$ $\Updownarrow$

(a) S contains all its bdry pts (i.e., $bS \subseteq S$)

$\Updownarrow$

Remark: (b) S^c is open

Why Remark ?

Justification of Remark :

① "(a) S contains its boundary (i.e., $bS \subseteq S$)"

$\Longrightarrow$ "(b) S^c is open"

To prove "S^c is open",

(we need to prove every $z_0 \in S^c$ is an interior pt of S^c)

Let $z_0 \in S^c$,

note: (1) z_0 cannot be an interior pt of S

(2) z_0 cannot be a bdry pt of S.

$\Longrightarrow$ z_0 must be an exterior pt of S

② "S^c is open" $\Longrightarrow$
(b)

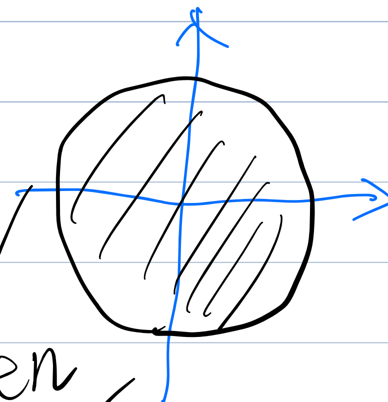
"S contains its boundary (i.e., $bS \subseteq S$)"
(a)

Ex.

E.g: ① $S=\{|z|\le 1\}$ is closed

why?

(a). $bS=\{|z|=1\}\subseteq S$



(b) $S^c=\{|z|>1\}$ open

② $S=\{|z|=1\}$ is closed.

(a) $bS=\{|z|=1\}\subseteq S$

(b) $S^c=\{|z|<1\}\cup\{|z|>1\}$ open

③ $S=\{|z|<1\}$ is NOT closed

why?

Ex

④ $S = \mathbb{C}$ is closed

why? E.x

Remark: $S = \mathbb{C}$ is open and closed.

Q: Is there any other set that is open and closed?

A:

Summarize:

① S is closed $\Leftrightarrow$ S^c is open

② S is open $\Leftrightarrow$ S^c is closed.

Def^n: The closure of $S \subseteq \mathbb{C}$ is defined as $\operatorname{cl} S = S \cup bS$.

Note:

E.g.: S