Chapter 1. Region in the complex plane

Let $S \subseteq \mathbb{C}$, $z_0 \in \mathbb{C}$.

z_0 must be in S

① z_0 is an interior pt of S if

$\exists$ a neighborhood $\{|z-z_0| < \varepsilon\}$ entirely contained in S

z_0 must be in S^c

② z_0 is an exterior pt of S if

$\exists$ $\{|z-z_0| < \varepsilon\}$ entirely contained in S^c

③ z_0 is a boundary pt of S if

z_0 is neither an interior pt nor an exterior pt of S

Remark: one and precisely one of the above ① ② ③ cases must occur.

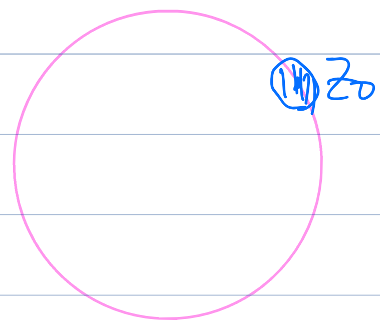
open and closed sets

- $S \subseteq \mathbb{C}$ is open
  $\iff$ every point in S is an interior pt of S
  ({interior pts of S} $= S$)

- $S \subseteq \mathbb{C}$ is closed
  $\iff$ S contains all its boundary pts (1)
  ($bS \subseteq S$)
  $\iff$ S^c is open. (2)

E.g 1: $S = \{|z| = 1\}$

Is S open? Is S closed?



Note: S has no interior pts:

$$\{\text{interior pts of } S\} = \phi$$

$\Rightarrow$ S is NOT open

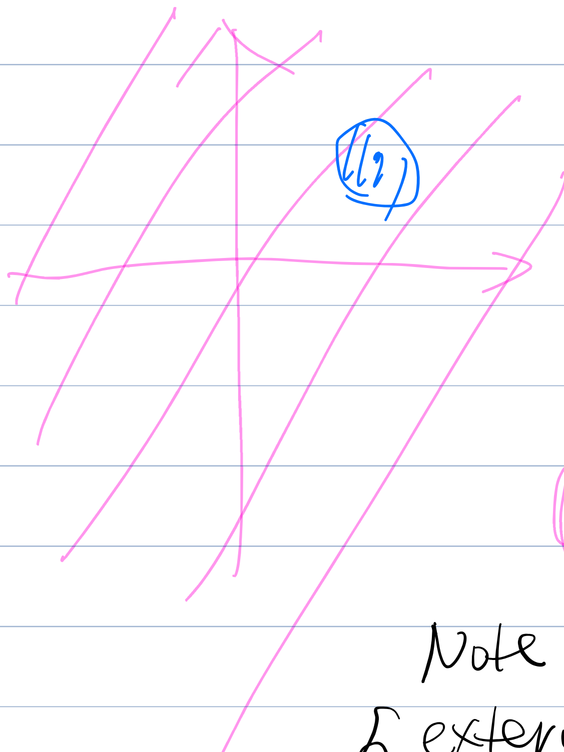
Note: $S^c = \{|z| < 1\} \cup \{|z| > 1\}$

$\Rightarrow$ S^c is open

$\Rightarrow$ S is closed.

E.g 2:

$$S = \mathbb{C}$$



every pt is an interior pt of S

$\Rightarrow$ S is open

$\{$interior pts of $S\} = \mathbb{C}$

Note $S^c = \phi$, $\Rightarrow$ $\{$exterior pts of $S\} = \phi$

$\partial S = \mathbb{C} - \{\text{interior}\} - \{\text{exterior}\} = \phi \subseteq S$

Remark: $S = \mathbb{C}$ is open and closed.

Q: Is there any other set that is open and closed?

A: ϕ is also open and closed.

why ϕ is closed? as $\phi^c = \mathbb{C}$ is open why $\mathbb{C}$ is closed? as $\mathbb{C}^c = \phi$ is open.

Summarize:

① S is closed $\iff$ S^c is open

② S is open $\iff$ S^c is closed.

(Note $(S^c)^c = S$)

Def^n: The closure of $S \subseteq \mathbb{C}$ is defined as $\text{cl}\, S = S \cup bS$.

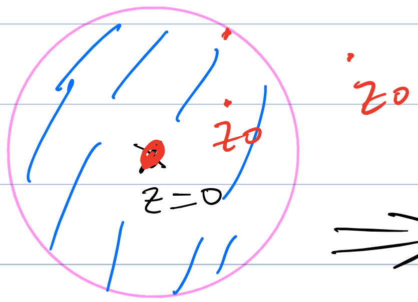
shorthand for closure

In some other books, they use "$\bar{S}$" for the closure of S.

Remark: for every $S \subseteq \mathbb{C}$, $\text{cl}\, S$ is always closed.

E.g: Let $S = \{0 < |z| < 1\}$. Find $\text{cl} S$.

Recall $\text{cl} S = S \cup bS$

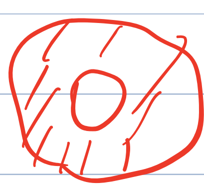


$bS = \{|z| = 1\} \cup \{0\}$.

$\Rightarrow$

$$\text{cl} S = S \cup bS$$
$$= \{|z| \leq 1\}$$

Ex: Let $S = \{1 < |z| < 2\}$. Find $\text{cl} S$



annulus with inner radius $= 1$

outer radius = 2

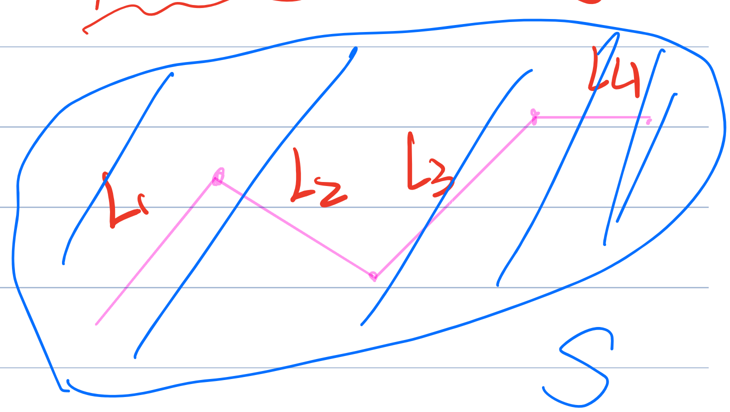
Connected sets, domains & regions

Def^n: An open set $S \subseteq \mathbb{C}$ is connected

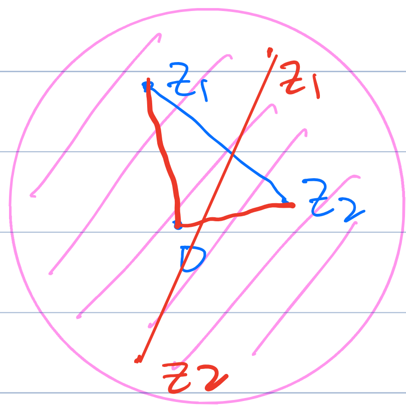
$$\Updownarrow$$

every two points z_1, z_2 in S can be joined by a polygonal line segment that is entirely contained in S

piecewise line segment

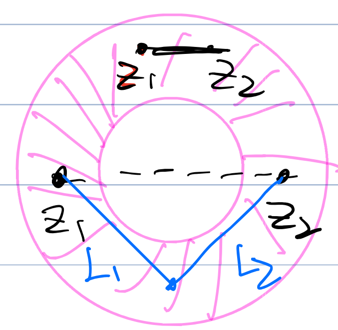


E.g: ① $\{|z|<1\}$ is connected



② $S = \{1 < |z| < 2\}$ annulus

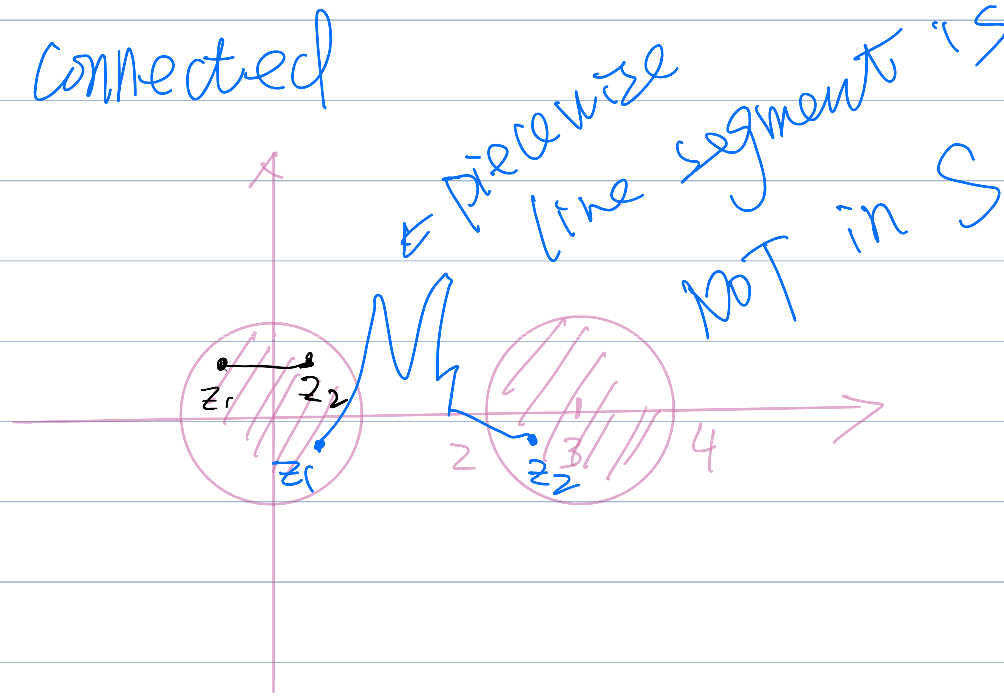
is connected.



③ $\{0<|z|<1\}$ is connected.

E.x.

④ $S = \{|z|<1\} \cup \{|z-3|<1\}$ is NOT connected



Def^n: A nonempty set $S \subseteq \mathbb{C}$ is called

(i) a domain $\Longleftrightarrow$ S is open and connected

(ii) a region $\Longleftrightarrow$ S is a domain together with some part of its boundary

can be $\emptyset$, a proper subset or the whole set of bS.

Note: a domain is always a region.

Remark: In Calculus ($\mathbb{R}$), we often consider functions $f: (a,b) \to \mathbb{R}$.

In complex analysis, we will often define functions f on either a domain or a region.

E.g.:

① $\{|z| < 1\}$ is open and connected

Hence it is a domain

② $$\underbrace{\{|z| \le 1\}}_{\text{NOT open}} = \underbrace{\{|z| < 1\}}_{S \text{ is a domain}} \cup \underbrace{\{|z| = 1\}}_{\partial S}$$

NOT open

Hence NOT a domain

S is a domain

∂S

is a

Hence $\{|z| \le 1\}$ is region

③ $\{0 < |z| \le 1\}$ is a region

is NOT open, Hence NOT a domain

$\{0 < |z| < 1\} \cup \{|z| = 1\}$

(a domain union with a proper subset of its boundary)

④ $\{|z| = 1\}$ NOT a domain, NOT a region.

E.x.

Def^n: $S \subseteq \mathbb{C}$ is bounded.

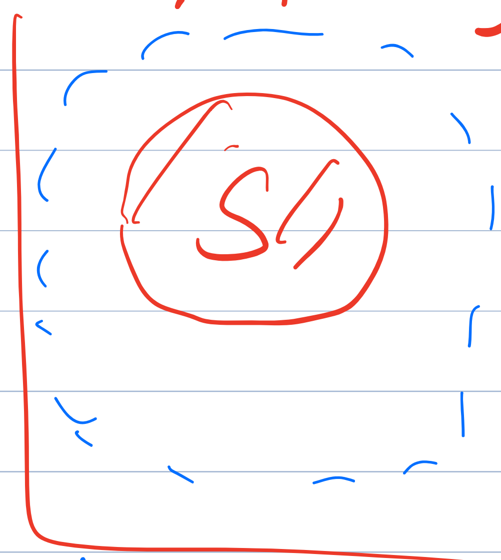
$\Updownarrow$

$S \subseteq \{|z| < R\}$ for some $R > 0$ finite

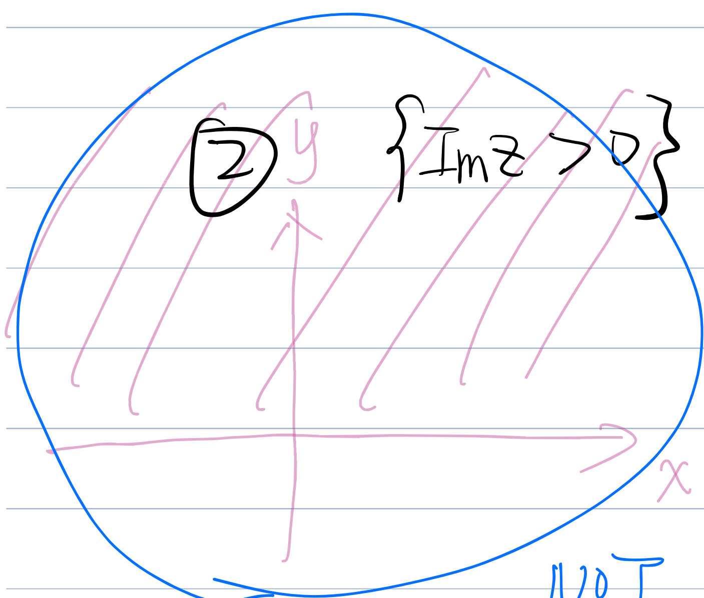
(i.e., you can draw a big circle such that S is inside the circle)

E.g.: ① $S = \{0 < |z| < 20\}$ is bdd

S is entirely inside $\{|z| = 100\}$



② $\{\operatorname{Im} z > 0\}$



NOT bdd

③ $\mathbb{C}$

NOT bdd

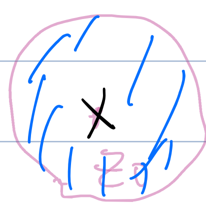
Recall: Let $z_0 \in \mathbb{C}$.

"a neighborhood of z_0" means

$\{|z-z_0| < \varepsilon\}$ for some $\varepsilon > 0$.

Def^n: (deleted neighborhood)

The set $\{0 < |z-z_0| < \varepsilon\}$ is called a deleted nbhd or a deleted ε-nbhd of z_0



means "with center z_0 deleted"

<u>Defⁿ</u>: Given $S \subseteq \mathbb{C}$, $z_0 \in \mathbb{C}$,

z_0 is an accumulation pt of

$S \iff$

every deleted nbhd of z_0

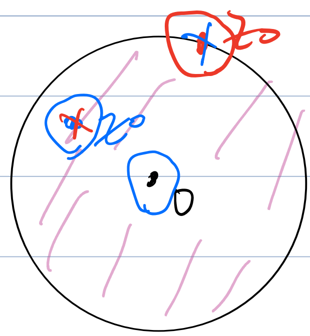
contains (at least one) points

of S

(That is, S has pts that are arbitrarily close to z_0, but NOT equal to z_0)

Eg: ① $S = \{0 < |z| < 1\}$

accumulation pts of S:



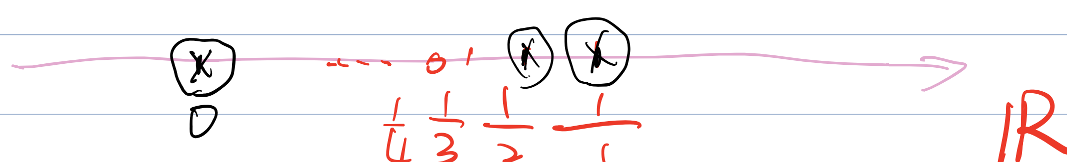
$\{|z|=1\} \cup \{|z|<1\}$

$= \{|z| \le 1\}$

$n = 1, 2, 3, 4, \ldots$

② $S = \{\frac{1}{n} : n \in \mathbb{Z}, n > 0\}$

accumulation pts of S: $\{0\}$



$$\lim_{n \to \infty} \frac{1}{n} = 0$$