Functions of a complex variable

Let $S \subseteq \mathbb{C}$

A function $f: S \to \mathbb{C}$ is an assignment/rule

$$\underset{\substack{\cap \\ S}}{z} \xrightarrow{f} w = f(z) \in \mathbb{C}$$

---

1 • We will call f a (complex) function defined on S

(f will be always assumed to take values in $\mathbb{C}$)

• f can be regarded as from $S \subseteq \mathbb{R}^2 \to \mathbb{R}^2$

$$\boxed{\begin{array}{c} \mathbb{C} \simeq \mathbb{R}^2 \\ x+iy \simeq (x,y) \end{array}}$$

For that, write:

$$z = x + iy \simeq (x,y) \in \mathbb{R}^2$$

$$w = u + iv \simeq (u,v) \in \mathbb{R}^2$$

and write

$$w = f(z) = u(x,y) + i\,v(x,y)$$

(i.e., $\underset{\substack{\cap \\ \mathbb{R}^2}}{(x,y)} \longrightarrow \underset{\substack{\cap \\ \mathbb{R}^2}}{(u(x,y),\, v(x,y))}$)

In the above, as before.

$$\begin{cases} u(x,y) = \mathrm{Re}\, f(z) & \text{real part} \\ v(x,y) = \mathrm{Im}\, f(z) & \text{imaginary part} \end{cases}$$

$z \to z^2$

E.g.: ① $f(z) = z^2$: $\mathbb{C} \to \mathbb{C}$ $(\mathbb{R}^2 \to \mathbb{R}^2)$

Find the corresponding u, v.

A: write $z = x + iy$

$$\Rightarrow f(z) = z^2 = (x+iy)^2 = (x+iy)(x+iy)$$
$$= (x^2 - y^2) + i(2xy)$$

Hence $u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$

$$f: (x,y) \to (x^2 - y^2, 2xy)$$

$z \to \frac{1}{z}$

② $f(z) = \frac{1}{z}$: $\{z \neq 0\} \to \mathbb{C}$ $(\mathbb{R}^2 \to \mathbb{R}^2)$

A: Write $z = x + iy \Rightarrow$

$$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{1}{x+iy} \frac{(x-iy)}{(x-iy)}$$

$$= \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right)$$

Hence, $u(x,y) = \frac{x}{x^2+y^2}$, $v(x,y) = \frac{-y}{x^2+y^2}$

Sometimes it is convenient to use polar coordinates. That is, when $z \neq 0$, write $z = re^{i\theta}$

$$f(z) = f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta)$$

$$f: (r,\theta) \in \mathbb{R}^2 \Rightarrow (u,v) \in \mathbb{R}^2$$

E.g.: $f(z) = \frac{1}{z}$: $\{z \neq 0\} \to \mathbb{C}$

A: Write $z = re^{i\theta}$, $\Rightarrow$

Hint: $\frac{1}{e^{i\theta}} = e^{-i\theta}$

$$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}\frac{1}{e^{i\theta}}$$

$$= \frac{1}{r}e^{-i\theta} = \frac{1}{r}e^{i(-\theta)}$$

$$= \frac{1}{r}(\cos(-\theta) + i\sin(-\theta))$$

$$= \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$= \frac{1}{r}\cos\theta + i\left(-\frac{1}{r}\sin\theta\right)$$

Hence $u(r,\theta) = \frac{1}{r}\cos\theta$, $v(r,\theta) = -\frac{1}{r}\sin\theta$

Functions as transformations

Sometimes it is helpful to regard a particular function as a transformation of the $\mathbb{C}$-plane (or part of the $\mathbb{C}$-plane)

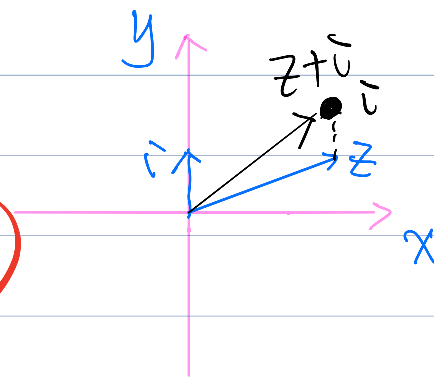
E.g what does f do to z?

translation

① $f(z) = z + i$ $\qquad z \xrightarrow{f} w = z + i$

$\boxed{i \sim (0,1)}$

f translates z by i (or $(0,1)$)

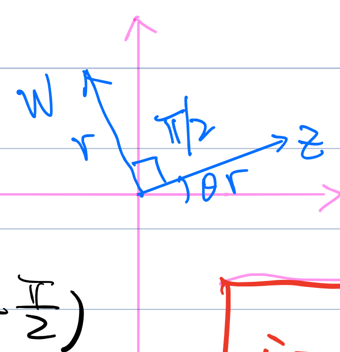


② $f(z) = iz$

write $z = re^{i\theta}$

$$\Rightarrow w = f(z) = iz = ire^{i\theta}$$

$$= e^{i\frac{\pi}{2}} re^{i\theta} = re^{i(\theta + \frac{\pi}{2})}$$



$\boxed{i = 1 \cdot e^{i\frac{\pi}{2}}}$

f = ratation by $\pi/2$

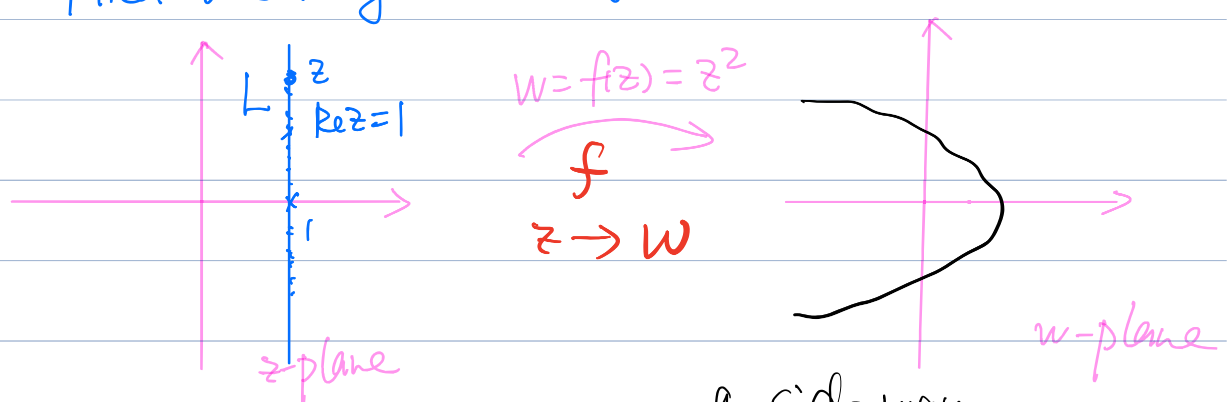
(counter-clockwise)

Functions as mappings

We can also think of f as a mappings from (part of) the "z-plane" to (part of) the "w-plane".

E.g (1) $f(z) = z^2$ $\qquad z \longrightarrow w = z^2$

Find the image under f of the line $\{\operatorname{Re} z = 1\}$.



a sideway parabola

$L = \{\operatorname{Re} z = 1\}$

A:

Note: $z \xrightarrow{f} w = z^2$

Let $z \in L = \{\operatorname{Re} z = 1\}$

Then $z = 1 + iy$, $y \in \mathbb{R}$

$$w = f(z) = z^2 = (1+iy)^2$$

$$= 1 - y^2 + i(2y)$$

Hence $w = u + iv \simeq (u, v)$

with $u = 1 - y^2$, $v = 2y$

on w-plane, consider all such

$$w = u + iv \simeq (u, v) = (1 - y^2, 2y)$$

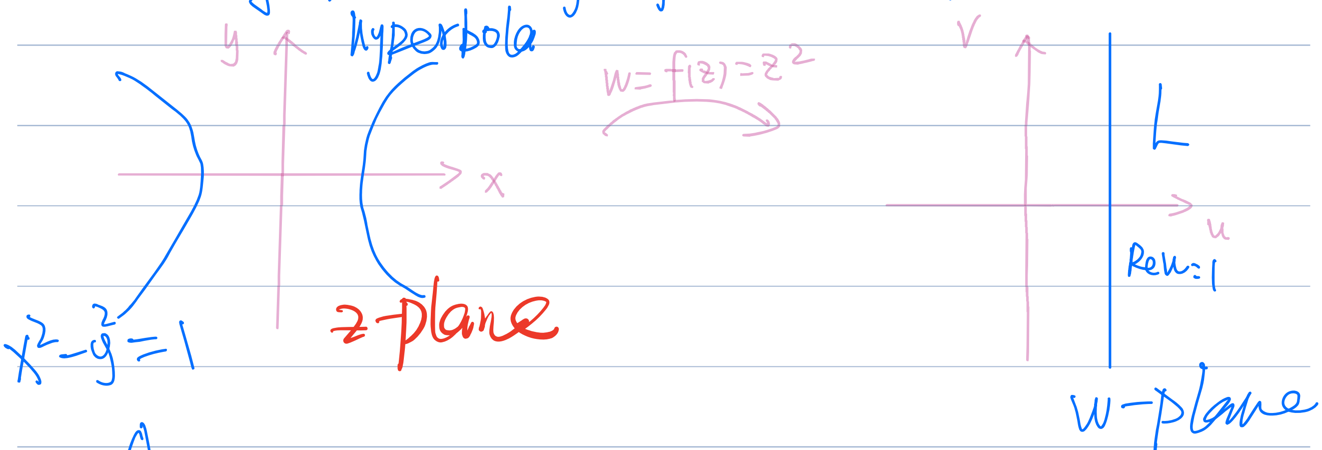
$y \in \mathbb{R}$

Hint: Can you find a relation between u, v?

$v = 2y \Rightarrow y = \frac{v}{2}$

$u = 1 - y^2 \Rightarrow u = 1 - \left(\frac{v}{2}\right)^2$, or $u = 1 - \frac{v^2}{4}$

(2) Let $f(z)=z^2$. Find the preimage (inverse image) under f of the line $\{\mathrm{Re}\, w=1\}$.



A: write $z=x+iy$

$$\Rightarrow \quad w=f(z)=z^2=(x+iy)^2$$

$$=\underbrace{(x^2-y^2)}_{u}+i\underbrace{(2xy)}_{v}$$

Q: What z gets mapped to $L=\{\mathrm{Re}\, w=1\}=\{u=1\}$?

A: $z=x+iy$ gets mapped to $L \Leftrightarrow u=x^2-y^2=1$

Limit and Continuity (from P44 of the book)

Q: How should we define limit in $\mathbb{C}$?

Let's first get some inspirations by recalling how 'limit' is defined in $\mathbb{R}$ (Calculus)

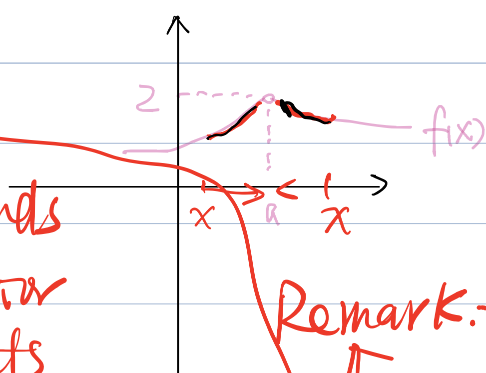
Recall in Calculus:

Let $f: I \to \mathbb{R}$ be a real function, and $I \subseteq \mathbb{R}$ is an interval containing the point $a \in \mathbb{R}$.

we say $\lim_{x \to a} f(x) = L$ if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$$

E.g:



In this case, $\lim_{x \to a} f(x) = 2$.

$\lim_{x \to a} f(x)$ depends on the behavior of f as x gets closer and closer to a

Remark: $\lim_{x \to a} f(x)$ does NOT depend on $f(a)$

but not equal a

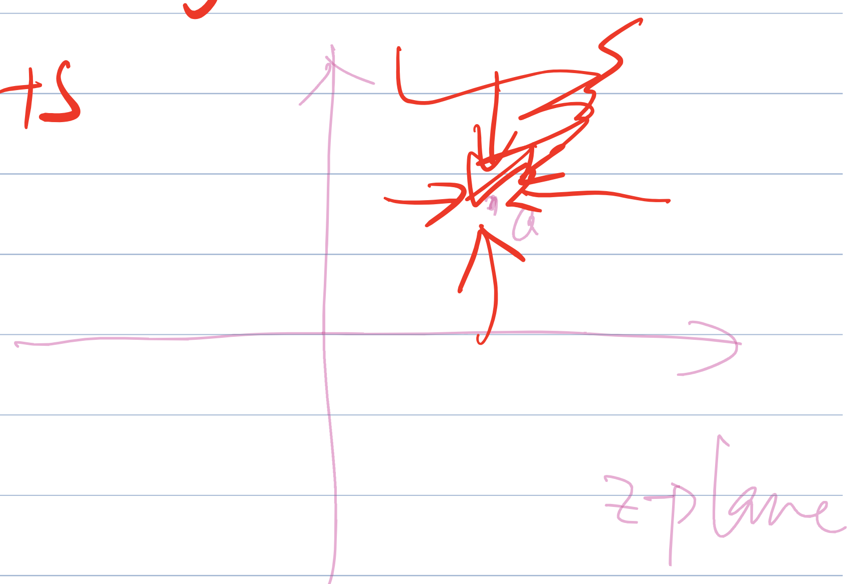
Remark: • In $\mathbb{R}$, x has 2 ways to approach a: from LHS and RHS

• limit at a does NOT depend at $f(a)$.

• The limit is unique!

Now in complex analysis, how should we define limit

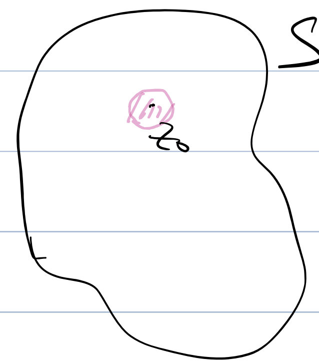
We cannot only look at LHS and RHS



Use $\varepsilon - \delta$ language!

Let $S \subseteq \mathbb{C}$, and $f: S \subseteq \mathbb{C} \to \mathbb{C}$ a (complex) function; let z_0 be an interior pt of S

<u>Def^n</u>: (Limit in complex analysis)



We say $\lim_{z \to z_0} f(z) = L \iff$

for every $\varepsilon > 0$, we can find $\delta > 0$ (depends on ε) such that

if $0 < |z - z_0| < \delta$, then $|f(z) - L| < \varepsilon$

Remark:

① It means when z gets close $\boxed{\text{to } z_0}$, but not equal to z_0, in whichever direction

$f(z)$ gets closer to L.

② The Limit is unique!

E.g 1. Find $\lim_{z \to z_0} f(z)$ and give a proof.

(a) $f(z) = 2iz$, $z_0 = 3$.

(b) $f(z) = \bar{z}$, $\quad z_0 = a \in \mathbb{C}$.

E.g 2 : Let $f(z) = \begin{cases} \dfrac{z}{\bar{z}} & z \in \mathbb{C} - \{0\} \\ 0 & z = 0 \end{cases}$

prove $\lim_{z \to 0} f(z)$ DNE