pg 61 8b) Suppose the limit $f'(z)$ exists at any $z = z_0$

$$\therefore \lim_{\Delta z \to 0} \frac{Im(z_0 + \Delta z) - Im(z_0)}{\Delta z} \text{ exists}$$

let $\Delta z = \Delta x$ approach along the Real axis

$$\therefore \lim_{\Delta x \to 0} \frac{Im(z_0 + \Delta x) - Im(z_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{Im(z_0) - Im(z_0)}{\Delta x} \quad (\text{since } \Delta x \text{ is real})$$

$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x}$$

$$= 0$$

let $\Delta z = i\Delta y$ approach along the imaginary axis

$$\therefore \lim_{\Delta y \to 0} \frac{Im(z_0 + i\Delta y) - Im(z_0)}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{Im(z_0) + \Delta y - Im(z_0)}{\Delta y} \quad (\text{since } i\Delta y \text{ is purely imaginary})$$

$$= \lim_{\Delta y \to 0} \frac{\Delta y}{\Delta y}$$

$$= 1$$

Since the limits don't match we get a contradiction pg 62-9.

$$\frac{f(0+\Delta z)-f(0)}{\Delta z} = \frac{\overline{\Delta z}^2 / \Delta z - 0}{\Delta z}$$

$$= \left(\frac{\overline{\Delta z}}{\Delta z}\right)^2$$

if Δz is real, say $\Delta z = \Delta x$

$$\lim_{\Delta z \to 0} \frac{f(0+\Delta z)-f(0)}{\Delta z} = \lim_{\Delta x \to 0} \left(\frac{\Delta x}{\Delta x}\right)^2$$

$$= 1$$

if Δz is real, say $\Delta z = i\Delta y$

$$\lim_{\Delta z \to 0} \frac{f(0+\Delta z)-f(0)}{\Delta z} = \lim_{\Delta y \to 0} \left(\frac{-i\Delta y}{i\Delta y}\right)^2$$

$$= \lim_{\Delta y \to 0} (-1)^2$$

$$= 1$$

if Δz is along $y=x$, say $\Delta z = \Delta x + i\Delta x$

$$\lim_{\Delta z \to 0} \frac{f(0+\Delta z)-f(0)}{\Delta z} = \lim_{\Delta x \to 0} \left(\frac{\Delta x - i\Delta x}{\Delta x + i\Delta x}\right)^2$$

$$= \lim_{\Delta x \to 0} \left(\frac{1-i}{1+i}\right)^2$$

$$= \frac{(1-i)^2}{(1+i)^2}$$

$$= \frac{-2i}{2i} = -1$$

Thus f cannot be differentiable at $z=0$

pg 70 1(c):

$f(z) = 2x + ixy^2$

$u(x,y) = 2x \qquad v(x,y) = xy^2$

$u_x = 2 \qquad v_x = y^2$

$u_y = 0 \qquad v_y = 2xy$

$\therefore$ f is differentiable at z

$\iff u_x = v_y$

$u_y = -v_x$

$\iff 2xy = 2$

$y^2 = 0$

This is not possible

$\therefore$ f is not differentiable at any point

pg 70 2(b):

$f(z) = e^{-x} e^{-iy} = e^{-x}(\cos y - i \sin y)$

$\therefore u(x,y) = e^{-x} \cos y$

$v(x,y) = -e^{-x} \sin y$

$u_x = -e^{-x} \cos y \qquad v_x = e^{-x} \sin y$

$u_y = -e^{-x} \sin y \qquad v_y = -e^{-x} \cos y$

$\therefore \quad u_x = v_y \qquad \Rightarrow \quad f$ is differentiable at every point

$\quad\;\; u_y = -v_x$

$$f'(z) = u_x + i v_x$$

$$= -e^{-x}\cos y + i e^{-x}\sin y$$

$$u_{xx} = e^{-x}\cos y \qquad v_{xx} = -e^{-x}\sin y$$

$$u_{xy} = e^{-x}\sin y \qquad v_{xy} = e^{-x}\cos y$$

$\therefore \quad u_{xx} = v_{xy} \qquad \Rightarrow \quad f'(z)$ is differentiable everywhere

$\quad\;\; u_{xy} = -v_{xx}$

$$f''(z) = u_{xx} + i v_{xx}$$

$$= e^{-x}\cos y - i e^{-x}\sin y$$

pg 71 3 (b)

$$f(z) = x^2 + i y^2$$

$$u(x,y) = x^2 \qquad v(x,y) = y^2$$

$$u_x = 2x \qquad v_x = 0$$

$$u_y = 0 \qquad v_y = 2y$$

$f(z)$ is differentiable $\Leftrightarrow$ $u_x = v_y \quad u_y = -v_x$

$\Leftrightarrow \quad x = y$

$$f'(x+ix) = u_x(x,x) + i\, v_x(x,x)$$

$$= 2x$$

p 71 3(c) :-

$$f(z) = z \operatorname{Im} z$$

$$= (x+iy)\,y$$

$$= xy + i y^2$$

$$u(x,y) = xy \qquad v(x,y) = y^2$$

$$u_x = y \qquad v_x = 0$$

$$u_y = x \qquad v_y = 2y$$

$f(z)$ is differentiable $\Leftrightarrow$ $u_x = v_y \quad u_y = -v_x$

$\Leftrightarrow \quad y = 0 \qquad x = 0$

$$f'(0) = u_x(0) + i\, v_x(0) = 0$$