1. We parametrize the contour $|z-1| = 2$ by setting

$z = 1+2e^{it}$ $\quad 0 \le t \le 2\pi$

$$\therefore \int_C \frac{1}{z-1}\,dz = \int_0^{2\pi} \frac{1}{1+2e^{it}-1}\, 2ie^{it}\,dt$$

$$= \int_0^{2\pi} \frac{1}{2e^{it}}\, 2ie^{it}\,dt$$

$$= \int_0^{2\pi} i\,dt$$

$$= 2\pi i$$

2. for $z \in \operatorname{Im} C$, $\quad |z| = 2$

$$\therefore \quad |z^2-1| \ge |z|^2 - |1| = |z|^2 - 1 = 4-1 = 3$$

$$\Rightarrow \quad \left|\frac{1}{z^2-1}\right| \le \frac{1}{3} \quad \text{for } z \in \operatorname{Im} C$$

$$L(C) = \frac{2\pi 2}{4} = \pi$$

$$\therefore \quad \left|\int_C \frac{1}{z^2-1}\,dz\right| \le \frac{\pi}{3}$$

3. For $z \in \operatorname{Im} C_R$, $\quad |z| = R$

$$\therefore \quad |2z^2-1| \le 2|z|^2 + 1 = 2R^2+1$$

$$|z^4+5z^2+4| = |(z^2+4)(z^2+1)|$$
$$\ge (|z|^2-4)(|z|^2-1)$$

Here we used the left hand side of the triangle inequality for each factor

$$\therefore \quad \left| \frac{2z^2-1}{z^4+5z^2+4} \right| \leq \frac{2R^2+1}{(R^2-4)(R^2-1)}$$

$$\ell(C) = \frac{2\pi R}{2} = \pi R$$

$$\therefore \quad \left| \int_{C_R} \frac{2z^2-1}{z^4+5z^2+4} \right| \leq \frac{2R^2+1}{(R^2-4)(R^2-1)} \cdot \pi R$$

4. a) Take $F(z) = \frac{z^3}{3}$

then $F'(z) = z^2$ on $\mathbb{C}$

$$\therefore \int_0^{1+i} z^2 \, dz = F(1+i) - F(0)$$

$$= \frac{(1+i)^3}{3}$$

$$= \frac{2i(1+i)}{3}$$

$$= \frac{2(-1+i)}{3}$$

b) Take $F(z) = 2\sin\left(\frac{z}{2}\right)$

$\therefore \quad F'(z) = \cos\left(\frac{z}{2}\right)$ on $\mathbb{C}$

$$\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right) dz = F(\pi+2i) - F(0)$$

$$= 2\sin\left(\frac{\pi}{2}+i\right)$$

$= 2\cos(i)$

$= e^{-1} + e^{1}$

c) take $F(z) = \frac{(z-2)^4}{4}$

$F'(z) = (z-2)^3$ on $\mathbb{C}$

$\therefore \int_1^3 (z-2)^3 \, dz = F(3) - F(1)$

$= \frac{1^4}{4} - \frac{(-1)^4}{4}$

$= 0$

5. let $F(z) = \frac{(z-4)^{-4}}{4}$ on $\mathbb{C} \setminus \{4\}$

then $F'(z) = (z-4)^{-5}$ on $\mathbb{C} \setminus \{4\}$

$\therefore \int_C \frac{1}{(z-4)^5} dz = 0$ since C is a closed contour in $\mathbb{C} \setminus \{4\}$.