In this chapter, we are going to focus on:

Q: Why/How do we define the real numbers?

We are not going to answer the question in the fullest detail, but we will discuss the motivation and the ways to define the real numbers, as well as the basic properties of \( \mathbb{R} \). This will be the starting point for proving propositions/theorems involving real numbers in later chapters.

To answer the question, we start from the set of natural numbers:

(1). \[ \mathbb{N} = \{1, 2, 3, \ldots \} \]

- Has addition & multiplication
  
  If \( a, b \in \mathbb{N} \), then \( a + b, ab \in \mathbb{N} \).

- No subtraction, No division.
  
  \[ x + 2 = 1 \text{ has no solution.} \]
  \[ \frac{1}{2} \notin \mathbb{N} \].

(2). The set of the integers:

\[ \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \} \]

- Has addition, multiplication, subtraction.

But still has no division:

Cannot solve \( 3x + 4 = 2 \) in \( \mathbb{Z} \).
This leads to the rational numbers:

\[ \mathbb{Q} = \left\{ \frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N} \right\}. \]

- Has addition, multiplication, subtraction and division

Q: Any problems with \( \mathbb{Q} \)?

1. Cannot solve \( x^2 = 2 \) in \( \mathbb{Q} \).

One cannot find \( h \in \mathbb{Q} \) such that \( h^2 = 2 \).

Let's give a proof to this fact.

Definition: A number is called an algebraic number if it satisfies a polynomial equation

\[ c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0 = 0. \]

where the coefficients \( c_0, c_1, \ldots, c_n \) are integers \( c_0 \neq 0 \) and \( n \geq 1 \).

E.g. 1. \( x = \frac{4}{17} \) is an algebraic number, since it satisfies \( 17x - 4 = 0 \).

More generally, all rational numbers are algebraic numbers.

2. \( x = \sqrt{2} \) is an algebraic number, since it satisfies \( x^2 - 2 = 0 \).

3. \( \sqrt[3]{2 + 3\sqrt{5}} \) is an algebraic number. Let

\[ x = \sqrt[3]{2 + 3\sqrt{5}}. \]

Then \( x^2 = 2 + 3\sqrt{5} \Rightarrow x^6 - 2 = 3x^2 \Rightarrow \]
\[ x = \sqrt[2]{2 + 3\sqrt{5}}. \text{ Then } x^2 = 2 + 3\sqrt{5} \Rightarrow \\
\] 
\[ x^2 - 2 = 3\sqrt{5}, \text{ thus } (x^2 - 2)^3 = 5 \]

Hence it satisfies
\[ x^6 + \ldots + 13 = 0 \]

İ.ş prove \( \sqrt[7]{\frac{4 - 2\sqrt{3}}{7}} \) is an algebraic number.

Thm (Rational Zeros theorem):

Suppose \( c_n, c_{n-1}, \ldots, c_0 \) are integers and \( r \) is a rational number satisfying the polynomial equation

\[ c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0 \quad (1) \]

Where \( n \geq 1, c_n \neq 0 \) and \( c_0 \neq 0 \). Let \( r = \frac{c}{d} \) where \( c, d \)
are integers having no common factors and \( d \neq 0 \).

Then \( c \) divides \( c_0 \) and \( d \) divides \( c_n \).

Remark:

1. In other words, the only rational candidates for solution of \( (1) \) have the form \( \frac{c}{d} \) where \( c \)
divides \( c_0 \) and \( d \) divides \( c_n \).

2. "\( a | b \)" is a shorthand for "\( a \) divides \( b \)"

Pf: Since \( r = \left( \frac{c}{d} \right) \) solves \( (1) \), we have

\[ c_n \left( \frac{c}{d} \right)^n + c_{n-1} \left( \frac{c}{d} \right)^{n-1} + \cdots + c_1 \left( \frac{c}{d} \right) + c_0 = 0. \]

Multiply by \( d^n \) to obtain

\[ c_n c^n + c_{n-1} c^{n-1}d + \cdots + c_1 c d^{n-1} + c_0 d^n = 0 \quad (2) \]

This implies

\[ r d^n + \cdots + c_{n-1} c d^{n-1} + c_0 d^n = 0 \]
This implies
\[ C \circ d^n = -C [C_n c^{n-1} + \cdots + C_1 d^{n-1}] \].

It follows that \( C \mid C \circ d^n \). But \( C \) and \( d^n \) have no common factors. \( \Rightarrow C \mid C_0 \).

Similarly, (2) also implies
\[ C_n C^n = -d [C_{n-1} C^{n-1} + C_{n-2} C^{n-2} d + \cdots + C_0 d^{n-1}] \].

It follows that \( d \mid C_n C^n \). But \( d \) and \( C^n \) have no common factors \( \Rightarrow d \mid C_0 \).

**Corollary:** Consider the polynomial equation
\[ x^n + C_{n-1} x^{n-1} + \cdots + C_1 x + C_0 = 0 \] (3)
where the coefficients \( C_0, \ldots, C_{n-1} \) are integers and \( C_0 \neq 0 \). Any rational solution of (3) must be an integer that divides \( C_0 \).

**Pf:** By the Thm, the denominator of \( r \) must divide the coefficient of \( x^n \), which is 1 in this case.

Thus \( r \in \mathbb{Z} \) and \( r \mid C_0 \).

E.g. \( \sqrt{2} \) is not a rational number.

**Pf:** Consider the polynomial equation \( x^2 - 2 \).

Note in this case, \( n = 2, C_2 = 1, C_1 = 0, C_0 = -2 \).

By Corollary, the rational soln (if exists) must be an integer that divides -2, thus can only be \( \pm 1, \pm 2 \). But none of them are
Can only be \( \pm 1, \pm 2 \). But none of them are solns of \( x^2 - 2 = 0 \). Hence \( x^2 - 2 = 0 \) has no rational solns, and \( \sqrt{2} \) is not rational.

Eg \( b = \frac{\sqrt{4 - 2\sqrt{3}}}{7} \) is not rational.

Pf: Way 1: Note \( b^2 = \frac{4 - 2\sqrt{3}}{7} \) ⇒ \( 7b^2 = 4 - 2\sqrt{3} \)

⇒ \( 2\sqrt{3} = 4 - 7b^2 \) ⇒ \( 12 = (7b^2 - 4)^2 \)

⇒ \( 49b^4 - 56b^2 + 4 = 0 \)

Hence \( b \) is a soln of \( 49x^4 - 56x^2 + 4 = 0 \).

By the Thm, the only possible rational solns are

\[ \pm 1, \pm \frac{1}{7}, \pm \frac{1}{49}, \pm 2, \pm \frac{2}{7}, \pm \frac{2}{49}, \pm 4, \pm \frac{4}{7}, \pm \frac{4}{49} \]

Compute and verify they are not solns.

Way 2: Suppose \( b \) is rational, then \( 7b^2 - 4 \) is also rational, so it is a rational soln to the eqn \( x^2 - 12 = 0 \). But by Corollary, the only possible rational solns to \( x^2 - 12 = 0 \) are \( \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12 \). It is easy to see they are not solns of \( x^2 - 12 = 0 \).

Hence \( x^2 - 12 = 0 \) has no rational solns.

This is a contradiction.
If we throw \( \sqrt{2}, \sqrt{3}, \ldots \) into \( \mathbb{Q} \), do we get \( \mathbb{R} \)?

The answer is No.

2. \( \mathbb{Q} \) is not complete. "\( \mathbb{Q} \) has holes/gaps"

E.g. \( \lim_{n \to \infty} (1 + \frac{1}{n})^n \) is not in \( \mathbb{Q} \)

E.g. Consider the graph of \( y = x^2 - 2 \).

It crosses the \( x \)-axis

But it slips all rational numbers.

If we fill in these holes/gaps, we get the real number set \( \mathbb{R} \).

There are three equivalent ways to define \( \mathbb{R} \):

1. Dedekind cuts:
   A real number is the same as a division of \( \mathbb{Q} \) into two "halves".

\[
\begin{align*}
\{ x \in \mathbb{Q} : x < 0 \text{ or } x^2 < 2 \} & \quad \{ x \in \mathbb{Q} : x > 0 \text{ and } x^2 > 2 \} \\
\end{align*}
\]

The "cut" gives a way to define \( \sqrt{2} \).

2. Cauchy sequences:

A sequence is "integral" on \( \mathbb{R} \).
2) Cauchy sequences:

A real number is the "limit" of a sequence of rational numbers which get closer and closer together.

E.g. \( e = \lim_{n \to \infty} (1 + \frac{1}{n})^n \).

3) Numbers with (possibly) infinite decimal expansions

\( \sqrt{2} = 1.414 \ldots \)

Next we discuss the axioms/key properties of \( \mathbb{R} \):

(1) The field axioms

(2) The order structure property

(3) The completeness axiom.

(1) The field axiom for \( \mathbb{R} \)

\( A_1 \): \( a + (b + c) = (a + b) + c \) for all \( a, b, c \)

\( A_2 \): \( a + b = b + a \) for all \( a, b \)

\( A_3 \): \( a + 0 = a \) for all \( a \in \mathbb{R} \).

\( A_4 \): For each \( a \), there is some \( -a \in \mathbb{R} \), such that

\( a + (-a) = 0. \)

\( M_1 \): \( a(bc) = (ab)c \) for all \( a, b, c \)

\( M_2 \): \( ab = ba \) for all \( a, b \)

\( M_3 \): \( a \cdot 1 = a \) for all \( a \).
M3: $a \cdot 1 = a$ \text{ for all } a.

M4: For each $a \neq 0$, there is some $a^{-1} \in \mathbb{R}$ such that $a \cdot a^{-1} = 1$.

DL: $a(b + c) = ab + ac$ \text{ for all } a, b, c.

(2) The order structure $\leq$

01: Given $a$ and $b$, either $a \leq b$ or $b \leq a$.

02: If $a \leq b$ and $b \leq a$, then $a = b$.

03: If $a \leq b$ and $b \leq c$, then $a \leq c$.

04: If $a \leq b$, then $a + c \leq b + c$.

05: If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

\textbf{Defn}: A field (meaning satisfies A1-DL) with an ordering satisfying 01-05 is called an ordered field.

$\Rightarrow$ $\mathbb{R}$ is an ordered field.

$\mathbb{Q}$ is an ordered field.

But $\mathbb{R}$ is a complete ordered field.

(3) The completeness Axiom

\textbf{Defn}. Let $S$ be a nonempty subset of $\mathbb{R}$

(a) The maximum of $S$, denoted by $\max S$, is the number so satisfying

$\forall x \in S$
the number $s$ satisfying

1. $s \in S$
2. $s \geq a$ for all $a \in S$.

If there is no such $s$, we say $\max S$ does not exist.

(b) The minimum of $S$, denoted by $\min S$, is

the number $s$ satisfying

1. $s \in S$
2. $s \leq a$ for all $a \in S$.

If there is no such $s$, we say $\min S$ does not exist.

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**Example**

1. $\max \{1, 2, 3, 4, 5\} = 5$, $\min \{1, 2, 3, 4, 5\} = 1$

2. Let $a, b \in \mathbb{R}$ and $a < b$. Recall

$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$; $(a, b) = \{x \in \mathbb{R} : a < x < b\}$

$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$; $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$.

$\max [a, b] = b$, $\min [a, b] = a$

$\max (a, b) \text{ D.N.E.} \quad \min (a, b) = a$

3. $\max \mathbb{Z} \text{ D.N.E.} \quad \min \mathbb{Z} \text{ D.N.E.}$

$\max \mathbb{N} \text{ D.N.E.} \quad \min \mathbb{N} = 1$.

4. Write $S = \{r \in \mathbb{Q} : 0 \leq r \leq \sqrt{2}\}$.

Then $\min S = 0$, $\max S \text{ D.N.E.}$

**Definition:** Let $S$ be a nonempty subset of $\mathbb{R}$.

(a) If a real number $M$ satisfies $s \leq M$ for
(a) If a real number \( M \) satisfies \( S \leq M \) for all \( s \in S \), then \( M \) is called an upper bound of \( S \) and the set \( S \) is said to be bounded above.

(b) If a real number \( m \) satisfies \( s \geq m \) for all \( s \in S \), then \( m \) is called a lower bound of \( S \) and \( S \) is said to be bounded below.

(c) The set \( S \) is said to be bounded if it is bounded above and bounded below.

E.g. 1. \( \mathbb{N} \) is bounded below but not bounded above.

2. \( \forall r \in \mathbb{Q}: 0 \leq r \leq \sqrt{2} \) is bounded.

Defn.: Let \( S \) be a nonempty subset of \( \mathbb{R} \).

(a) If \( S \) is bounded above and \( S \) has a least upper bound \( M_0 \in \mathbb{R} \), then we call it the supremum of \( S \).

Notation: \( \sup S = M_0 \)

(b) If \( S \) is bounded below and \( S \) has a greatest lower bound \( m_0 \in \mathbb{R} \), then we call it the infimum of \( S \).

Notation: \( \inf S = m_0 \).

E.g. 1. If \( a, b \in \mathbb{R} \) and \( a < b \), then

\[
\sup [a, b] = \sup (a, b) = \sup [a, b] = \sup (a, b) = b.
\]

2. \( \inf \mathbb{N} = 1 \quad \sup \mathbb{N} = +\infty \).
3. Let $A = \{ r \in \mathbb{Q} : 0 \leq r \leq \sqrt{2} \}$. Then $\sup A = \sqrt{2}$ and $\inf A = 0$.

4. Let $A = \{ \frac{1}{n^2} : n \in \mathbb{N} \text{ and } n \geq 3 \}$. Then $A$ is bounded.
   $\sup A = \frac{1}{9}$, $\inf A = 0$.

Completeness Axiom
Every nonempty subset $S$ of $\mathbb{R}$ that is bounded above has a least upper bound. In other words, $\sup S$ exists and is a real number.

Corollary 4.5. Every nonempty subset $S$ of $\mathbb{R}$ that is bounded below has a greatest lower bound $\inf S$.

Archimedean property
If $a > 0$ and $b > 0$, then for some integer $n$, we have $na > b$.

Density of $\mathbb{Q}$
If $a, b \in \mathbb{R}$ and $a < b$, then there is $r \in \mathbb{Q}$ such that $a < r < b$.

Remark: Some notations we will use in the future.
$c = \lim_{x \to \infty} \frac{x^2 + 1}{x - 10}$, $a < x$.
Remark: Some notations we will use ……

$[a, \infty) = \{ x \in \mathbb{R} : a \leq x \}$, $(a, +\infty) = \{ x \in \mathbb{R} : a < x \}$.

$(-\infty, b] = \{ x \in \mathbb{R} : x \leq b \}$, $(-\infty, b) = \{ x \in \mathbb{R} : x < b \}$. 