Chapter 2.4

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\[ \text{Cauchy Sequence} \]

**Defn:** A sequence \( \{s_n\} \) is called a Cauchy sequence if for each \( \varepsilon > 0 \), \( \exists N \in \mathbb{N} \) such that
\[ m, n > N \Rightarrow |s_n - s_m| < \varepsilon. \]

**Lemma 10.9**

Convergent sequences are Cauchy sequences.

**Pf.** Let \( \{s_n\} \) be a convergent sequence. Write \( L := \lim s_n. \)

Let \( \varepsilon > 0. \) Then by defn of limit, \( \exists N \in \mathbb{N} \) s.t.
\[ n > N \Rightarrow |s_n - L| < \frac{\varepsilon}{2}. \]

We may also write
\[ m > N \Rightarrow |s_m - L| < \frac{\varepsilon}{2}. \]

So \( m, n > N \Rightarrow |s_n - s_m| \leq |s_n - L| + |L - s_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \)

Thus \( \{s_n\} \) is a Cauchy sequence.

**Lemma 10.10**

Cauchy sequences are bounded.

**Pf.** Assume \( s_n \to L. \)

Choose \( \varepsilon = 1. \) Then \( \exists N \in \mathbb{N} \) s.t.
\[ m, n > N \Rightarrow |s_n - s_m| < 1. \]
Choose \( r = 1 \). \( \cdots \)

\[ m, n > N \implies |S_n - S_m| < 1. \]

Fix \( N_0 \in \mathbb{N} \) s.t. \( N_0 > N \). Then

\[ n \geq N_0 \implies n > N \implies |S_n - S_{N_0}| < 1 \]

By triangle inequality,

\[ |S_n| - |S_{N_0}| \leq |S_n - S_{N_0}| < 1 \]

Thus \( |S_n| < |S_{N_0}| + 1 \) if \( n \geq N_0 \).

Let \( M = \max \{ |S_1|, \ldots, |S_{N_0}| - 1, |S_{N_0} + 1| \} \).

Then \( |S_n| \leq M, \forall n \in \mathbb{N} \).

**Thm 10.11**

\( \{S_n\} \) is a convergent sequence \( \iff \{S_n\} \) is a Cauchy sequence.

**Pf:**

1. “\( \Rightarrow \)”
   
   This part is already proved in Lemma 10.9.

2. “\( \Leftarrow \)” Assume \( \{S_n\}\) is a Cauchy sequence
   
   By Thm 10.7, we only need to prove
   
   \[ \liminf S_n = \limsup S_n. \quad (1) \]

Let \( \varepsilon > 0 \). Since \( \{S_n\} \) is a Cauchy sequence,

\[ \exists N \in \mathbb{N} \text{ s.t. } m, n > N \implies |S_n - S_m| < \varepsilon. \]

In particular,

\[ S_n < S_m + \varepsilon, \forall n, m > N. \]

Thus \( S_n + \varepsilon \) is an upper bound for \( \{S_n : n > N\} \).

\[ \implies V_N = \sup \{S_n : n > N\} \leq S_n + \varepsilon, \forall m > N. \]

In this situation, \( \varepsilon < 0 \). \( \forall m > N. \)
\[ \Rightarrow V_N = \sup \{ S_n : n > N \} \leq s_{m+2} \cdot 3^{m/N}. \]

Note this implies \[ V_N - 2 \leq S_m \quad \forall m > N. \]

Hence \[ V_N - 3 \leq \inf \{ S_m : m > N \} = u_N. \] Thus

\[ \limsup S_n \leq V_N \leq u_N + 3 \leq \liminf S_n + 3. \]

\[ \Rightarrow \limsup S_n \leq \liminf S_n + 3 \]

But this holds for all \( \varepsilon > 0 \), we have \( \limsup S_n \leq \liminf S_n \).

The opposite inequality always holds, so we have established (1).