\section{18 properties of Continuous Functions}

\textbf{Defn:} Let $D \subseteq \mathbb{R}$ nonempty, $f: D \rightarrow \mathbb{R}$ a function.

We say $f$ is bounded on $D$ if $\exists M > 0$ s.t.

$$|f(x)| \leq M \text{ for all } x \in D.$$  

\textbf{Thm 18.1} Let $f$ be a continuous function from $[a,b]$ to $\mathbb{R}$.

Then

1. $f$ is bounded on $[a,b]$.

2. The function $f$ achieves its maximum and minimum values on $[a,b]$. That is,

$$\exists x_0, y_0 \in [a,b] \text{ s.t.}$$

$$f(x_0) \leq f(x) \leq f(y_0) \text{ for all } x \in [a,b].$$

\textbf{Pf:} Assume $f$ is not bounded on $[a,b]$. Then for each $n \in \mathbb{N}$, $n$ is not an upper bound for $|f|$ and thus

$$\exists \text{ some } x_n \in [a,b] \text{ s.t. } |f(x_n)| > n.$$  

Consider $(x_n)_{n=1}^{\infty}$. By Bolzano-Weierstrass Thm, $(x_n)$ has a subsequence $(x_{n_k})$ that converges to some real number $x_0$

\textbf{Claim:} $\lim x_{n_k} = x_0 \in [a,b].$

\textbf{Pf of claim:} Since $a \leq x_{n_k} \leq b$ for all $k$

$$\Rightarrow a \leq \lim x_{n_k} \leq b.$$  

Since $x_0 \in [a,b]$ and $f(x)$ is continuous in $[a,b]$

$$\Rightarrow f \text{ is continuous at } x_0.$$
Since \( x_0 \in [a, b] \) and \( f(x) \) is continuous in \([a, b]\)

\[ \Rightarrow f \text{ is continuous at } x_0 \]

Hence \( f(x_n) \to f(x_0) \)

But \( |f(x_n)| > n_k > k \Rightarrow \lim_{k \to +\infty} |f(x_n)| = +\infty \)

This is a contradiction. Thus \( f \) is bounded.

\( \Xi \) Consider \( \{f(x) : x \in [a, b]\} \). This is a bounded set.

By Completeness axiom, \( \sup \{f(x) : x \in [a, b]\} \) exists

Write \( M = \sup \{f(x) : x \in [a, b]\} < +\infty \).

Claim: \( \exists (y_n) \text{ in } [a, b] \text{ s.t. } f(y_n) \to M. \)

pf of Claim: Since \( M - \frac{1}{n} \) is NOT an upper bound.

\[ \Rightarrow \exists y_n \in [a, b] \text{ s.t. } f(y_n) > M - \frac{1}{n} \]

Note \( M - \frac{1}{n} < f(y_n) < M \Rightarrow f(y_n) \to M. \)

By Bolzano–Weierstrass thm, \( \exists \) a subsequence \( \{y_{nk}\} \) of

\( \{y_n\} \text{ s.t. } y_{nk} \to y_0. \) As before, we know \( y_0 \in [a, b]. \)

Since \( f \) is continuous at \( y_0 \), \( \Rightarrow f(y_0) = \lim f(y_{nk}) \Rightarrow M. \)

This proves the existence of \( y_0 \) in \( \Xi \)

Ex: prove the existence of \( x_0. \)

Remark: It is important that \( \text{dom}(f) \) is a closed interval \([a, b]\).

E.g. 1. \( f(x) = \frac{1}{x} \) on \((0, 1)\). Then \( f \) is NOT bdd, has no maximum/minimum on \((0, 1)\).

E.g. 2. \( f(x) = x^2 \) on \((-1, 1)\). Then \( f \) is bdd, but \( f \)

\[ \text{maximum } \leq (0, 1) \]
Thm 18.2 (Intermediate Value Theorem, I.V.T.)

If \( f \) is a continuous function on an interval \( I \).
Assume \( a, b \in I, a < b \) and \( f(a) \neq f(b) \).
Then for any \( y \) between \( f(a) \) and \( f(b) \), \( \exists c \in (a, b) \) s.t
\[ f(c) = y. \]

**Pf:** We will suppose \( f(a) < f(b) \) (The Case \( f(a) > f(b) \) is similar).

Let \( f(a) < y < f(b) \). Set \( S = \{ x \in [a, b] : f(x) < y \} \).

Then note \( a \in S, \Rightarrow S \neq \emptyset \); \( b \notin S \).
Write \( x_0 = \text{sup } S \) (why exists?) Then \( x_0 \in [a, b] \).

Since \( x_0 - \frac{1}{n} \) is not an upper bound for \( S \)
\[ \Rightarrow \exists S_n \subset S \text{ s.t } x_0 - \frac{1}{n} < S_n \leq x_0. \]

Then \( S_n \to x_0 \), and \( f(S_n) \to y \)

By continuity of \( f \) at \( x_0 \), \( f(x_0) = \lim f(S_n) \leq y \) (1)

Let \( t_n = \min \{ b, x_0 + \frac{1}{n} \} \). Since \( x_0 \leq t_n \leq x_0 + \frac{1}{n} \)
\[ \Rightarrow t_n \to x_0 \]

Therefore, by continuity of \( f \) at \( x_0 \), \( \Rightarrow \)
\[ f(x_0) = \lim f(t_n) \]

But \( t_n \notin S \) (why?), \( \Rightarrow f(t_n) \geq y, \forall n \)
\[ \Rightarrow f(x_0) = \lim f(t_n) \geq y \] (2)
By (1), (2), \( f(x_0) = y \).

**Corollary 18.3.** If \( f \) is a continuous function on an interval \( I \), then \( f(I) = \{ f(x) : x \in I \} \) is also an interval or a single pt.

**Pf:** If \( f \) is a constant function, then \( f(I) \) is a single pt.

Now assume \( f \) is not constant. By I. V. T \( \Rightarrow \)

The set \( J = f(I) \) has the property

If \( y_0 < y_1 \) in \( J \) and \( y_0 < y < y_1 \) \( \Rightarrow \) \( y \in J \) \( (*) \)

Note \( \inf J < \sup J \). (They might be \(-\infty, +\infty\))

**Claim:** For all \( \inf J < y < \sup J \), \( y \in J \).

**Pf:** Since \( \inf J < y < \sup J \), \( \exists y_0, y_1 \in J \),

s.t. \( y_0 < y < y_1 \). Hence \( y \in J \)

Thus \( J \) must be one of \((\inf J, \sup J)\), \((\inf J, \sup J]\), \([\inf J, \sup J)\), \([\inf J, \sup J]\).

**E.g.** Let \( f \) be a continuous function from \([0,1]\) to \([0,1]\).

prove \( f \) has a fixed pt in \([0,1]\). (That is, \( \exists x_0 \in [0,1] \), s.t. \( f(x_0) = x_0 \)).

**Pf:** Let \( g(x) = f(x) - x \). Then \( g(x_0) = f(x_0) - x_0 \geq 0 \)
\[ Pf: \] Let \( g(x) = f(x) - x \). Then \[ \int g' = \int (f' - 1) = f(x) - x \leq 0. \]

**Case I.** If \( g(0) = 0 \) or \( g(1) = 1 \), then \( f(x) \) has a fixed pt at 0 or 1.

**Case II.** If \( g(0) > 0 \) and \( g(1) < 0 \), then by I.V.T.

\[ \Rightarrow \exists \text{ some } 0 < x_0 < 1. \text{ s.t. } g(x_0) = 0. \]

This implies \( f(x_0) = x_0 \)

**E.g.** Let \( n > 1 \) be an odd positive integer.

\[ f(x) = x^n + x + 1 \]

prove \( \exists x_0 \in \mathbb{R} \) s.t. \( f(x_0) = 0. \)

**Pf.** Note \( f(-1) = -1, f(1) = 1 \). Thus by I.V.T., \( \exists -1 < x_0 < 1. \text{ s.t. } f(x_0) = 0. \)

**E.x:** Read Thm 18.4 - 18.6.