§19 Uniform Continuity part I.

Recall one version of the defn of continuity:

Let \( f : D \to \mathbb{R} \) be a function. We say \( f \) is continuous in \( D \), if at every point \( x_0 \in D \), for \( \forall \varepsilon > 0 \), there exists \( \delta > 0 \) s.t \( y \in D \) and \( |y - x_0| < \delta \Rightarrow |f(y) - f(x_0)| < \varepsilon \).

Remark: Sometimes we can choose a "\( \delta \)" that is independent of \( x_0 \). That is, we can some uniform "\( \delta \)" that only depends on \( \varepsilon \), and works for all \( x_0 \in D \). Sometimes we cannot find such a "uniform \( \delta \)".

E.g. Let \( f : \mathbb{R} \to \mathbb{R} \) be \( f(x) = 2x \). Prove \( f \) is continuous on \( \mathbb{R} \) and see whether we can find such a "uniform \( \delta \)."

Pf: Fix \( x_0 \in \mathbb{R} \). We will prove \( f \) is continuous at \( x_0 \).

Let \( \varepsilon > 0 \). Choose \( \delta = \frac{\varepsilon}{2} \) (Note this "\( \delta \)" only depends on \( \varepsilon \)). Then \( |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| = |2x - 2x_0| = 2|x - x_0| < 2\delta = \varepsilon \).

Hence we have found a "uniform \( \delta \)" that works for all \( x_0 \).
E.g. Let \( f(x) = \frac{1}{x^2} : (0, +\infty) \to \mathbb{R} \).

prove \( f(x) \) is continuous on \((0, +\infty)\) and check whether we can find a "uniform\( \delta\)".

Idea of \( pf.\): Given \( \varepsilon > 0 \), we need to choose \( \delta \) s.t.

\[
|x-x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.
\]

Note

\[
|f(x) - f(x_0)| = \left| \frac{1}{x^2} - \frac{1}{x_0^2} \right| = \frac{|x_0^2 - x^2|}{|x^2x_0^2|} = \frac{|x_0 - x|}{x^2x_0^2} \quad (1)
\]

Suppose \( \delta \leq \frac{x_0}{2} \). Then \( |x - x_0| < \delta \implies |x - x_0| < \frac{x_0}{2} \).

This implies

\[
|x| \geq |x_0| - |x - x_0| > \frac{x_0}{2}.
\]

\[
|x| \leq |x_0| + |x - x_0| < \frac{3x_0}{2}.
\]

\[
\implies |x_0 + x| = x_0 + x < \frac{5x_0}{2}.
\]

Hence by (1),

\[
|f(x) - f(x_0)| \leq \frac{|x_0 - x| \frac{5x_0}{2}}{x_0^2x_0^2} = \frac{10|x_0 - x|}{x_0^2} < \frac{10\delta}{x_0^3}.
\]

Thus we need \( \frac{10\delta}{x_0^3} \leq \varepsilon \implies \delta \leq \frac{x_0^3\varepsilon}{10} \).

Hence we can take \( \delta = \min \left\{ \frac{x_0}{2}, \frac{x_0^3\varepsilon}{10} \right\} \).
Formal pf: Exercise.

Note: In this example, $\delta$ depends on $x_0$. More precisely, for different values of $x_0 \in (0, +\infty)$, the value of $\delta = \min \{ \frac{x_0}{2}, \frac{x_0^2}{10} \}$ is different.

Q: If we work harder or use a better method, can we find a “uniform $\delta$” for $f(x) = \frac{1}{x^2}$: $(0, +\infty) \rightarrow \mathbb{R}$?

A: we will prove the answer is “no”. Indeed, we will show one can never find a “uniform $\delta$” for this function.

Before giving a detailed pf, we first introduce the following notion of uniform continuity.

**Defn (19.1):**

Let $f: D \rightarrow \mathbb{R}$ be a function. We say $f$ is uniformly continuous on $D$ if

for each $\varepsilon > 0$ there exists $\delta > 0$ s.t.

$$
\forall x, y \in D \quad |y - x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon .
$$

Sometimes we simply say that $f$ is uniformly continuous in this sense it is understood as that $f$ is uniformly
sometimes we simply say that \( f \) is uniformly continuous. In this case, it is understood as that \( f \) is uniformly continuous in \( \text{dom}(f) \).

Remark: \( f \) is uniformly continuous in \( D \) \iff we can find a "uniform \( \delta \)" that works for all \( x_0 \in D \).

E.g. prove \( f(x) = 2x \): \( \mathbb{R} \to \mathbb{R} \) is uniformly continuous.

Pf.: Let \( \varepsilon > 0 \). Choose \( \delta = \frac{\varepsilon}{2} \). Then

\[ |x - y| < \delta \Rightarrow |f(x) - f(y)| = 2|x - y| < 2\delta = \varepsilon. \]

Thus \( f(x) \) is uniformly continuous on \( \mathbb{R} \).

Remark: Whether the function is uniformly continuous not only depends on \( f \), but also depends on \( D \).

We will show that

\( f(x) = \frac{1}{x^2} \) is uniformly continuous on \( [a, +\infty) \) for any fixed \( a > 0 \).

\( f(x) = \frac{1}{x^2} \) is not uniformly continuous on \( (0, 1) \) and thus uniformly continuous on \( (0, +\infty) \).

E.g. prove \( f(x) = \frac{1}{x^2} \) is uniformly continuous on \( [a, +\infty) \) for \( a > 0 \).

Idea: Given \( \varepsilon > 0 \), we need to find \( \delta > 0 \) s.t

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Idea: Given $\varepsilon > 0$, we need to find $\delta > 0$ s.t.

$$x, y \in [a, +\infty), \ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.\$$

Note: $|f(x) - f(y)| = \frac{|y-x| |y+x|}{x^2 y^2}$\hfill (2)

Note: $x, y \in [a, +\infty) \Rightarrow$

$$\frac{|y+x|}{x^2 y^2} = \frac{\frac{y+x}{x^2 y^2}}{x^2 y^2} = \frac{1}{x^2 y^2} + \frac{1}{x y^2} \leq \frac{1}{a^3} + \frac{1}{a^3} = \frac{2}{a^3}$$

By (2) $\Rightarrow |f(x) - f(y)| \leq \frac{2}{a^3} \ |x-y| \leq \frac{2}{a^3} \delta$

We need to make $\frac{2}{a^3} \delta \leq \varepsilon \ \text{i.e.} \ \delta \leq \frac{a^3 \varepsilon}{2}$.

Formal pf: Let $\varepsilon > 0$. Choose $\delta = \frac{a^3 \varepsilon}{2}$. Then

$x, y \in [a, +\infty)$ and $|x-y| < \delta \Rightarrow$

$$|f(x) - f(y)| = \left| \frac{1}{x^2} - \frac{1}{y^2} \right| = \frac{|y-x| |y+x|}{x^2 y^2} = \frac{|y-x| (\frac{1}{x^2 y} + \frac{1}{x y^2})}{x^2 y^2} \leq \frac{|y-x|}{a^3} \frac{2}{a^3} \delta = \varepsilon.$$

Hence $f$ is uniformly continuous on $[a, +\infty)$

E.g. prove $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, 1)$

$\Rightarrow f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, +\infty)$.
Pf: We will prove the negation of (x) for \( f(x) \) on \((0, 1)\):

\[
\exists \varepsilon > 0, \text{ s.t. } \forall \delta > 0, \exists x, y \in (0, 1) \text{ satisfy } \\
| x - y | < \delta \quad \text{and} \quad | f(x) - f(y) | \geq \varepsilon .
\]

Idea: — we need to first find \( \varepsilon_0 > 0 \)

— given any \( \delta > 0 \), we need to find \( x, y \) s.t.

\( x, y \in (0, 1), \; | x - y | < \delta, \; | f(x) - f(y) | \geq \varepsilon_0 . \)

Hint: Choose \( \varepsilon_0 = 1 \) (or any smaller positive number).

Let \( x = \frac{1}{m}, \; y = \frac{1}{m+1}, \; m \text{ is some large integer.} \)

Note \( | x - y | = \frac{1}{m} - \frac{1}{m+1} = \frac{1}{m(m+1)} \to 0 \text{ as } m \to +\infty \)

But \( | f(x) - f(y) | = (m+1)^2 - m^2 = 2m+1 \geq 1 . \)

Formal pf: Choose \( \varepsilon = 1 \). Given any \( \delta > 0 \), we can find

some large \( m_0 \in N \text{ s.t. } \frac{1}{m_0(m_0+1)} < \delta . \)

Let \( x = \frac{1}{m_0}, \; y = \frac{1}{m_0+1} . \) Then

(1) \( x, y \in (0, 1) \)

(2) \( | x - y | = \frac{1}{m_0(m_0+1)} < \delta . \)

(3) \( | f(x) - f(y) | = (m_0+1)^2 - m_0^2 = 2m_0+1 > \varepsilon = 1 . \)

Hence \( f \) is not uniformly continuous on \((0, 1)\).
Hence \( f \) is not uniformly continuous on \((0, 1)\).

**Thm (19.4)**

If \( f \) is uniformly continuous on \( D \) and \((s_n)_{n=1}^{\infty} \) is a Cauchy sequence in \( D \), then \((f(s_n))\) is also a Cauchy sequence.

(That is, if \((s_n)_{n=1}^{\infty} \) is convergent

\[ \Rightarrow (f(s_n)) \text{ is also convergent} \]

**Pf:** Read P146.

This gives an easier way to prove \( f(x) = \frac{1}{x^2} \) is not uniformly continuous on \((0, 1)\).

**Pf:** Let \( s_n = \frac{1}{n+1}, \ n \geq 1 \). Then

1. \((s_n)_{n=1}^{\infty} \subseteq (0, 1)\)
2. \( \lim s_n = 0 \) (thus \( s_n \) is Cauchy)

But

3. \( f(s_n) = (n+1)^2 \) diverges to \( +\infty \).

Hence \( f \) is not uniformly continuous on \((0, 1)\).