Problem. 14.1 (d)

Proof. Note that $n - k \geq n/2$ for $n \geq 2k$. Thus we have $n - 1 \geq n/2, n - 2 \geq n/2$ and $n - 3 \geq n/2$ simultaneously for $n \geq 6$, so we have $a_n = \frac{n!}{n^{n+1}} \geq \frac{n!}{2^{n+1}} \geq \frac{(n-1)!}{2^{n+2}} \geq \frac{(n-2)!}{8^n} \geq \frac{(n-3)!}{16^n}$ for $n \geq 6$.

So $a_n$ does not converge to 0. That is to say, the series cannot converge. \hfill \Box

Problem. 14.2 (a)

Proof. Note that $a_n = \frac{n-1}{n^2} \geq \frac{n}{2n^2} = \frac{1}{2n}$ for all $n \geq 2$ (similar reason as previous question). That is to say, by the comparison test, the series diverge to infinite. \hfill \Box

Problem. 15.2 (a)

Proof. Consider $a_n = [\sin(\frac{n\pi}{6})]^n$, and the subsequence $n_k = 12k + 3$, we have $a_{n_k} = 1$, so $a_n$ does not converge to 0, which means the series cannot converge. \hfill \Box

Problem. 15.3

Proof. Consider the map $f(x) = \frac{1}{x \log x}$, Note that $f$ is positive decreasing on $[2, \infty)$. Using the integral test, the series, $\sum \frac{1}{n \log n}$ converges if and only if $\int_2^\infty f(x)dx < \infty$. Note that

$$
\int_2^N f(x)dx = \int_2^N \frac{1}{x \log x} dx
= \int_2^N \frac{1}{(\log x)^p} d\log x
= \begin{cases} 
\frac{1}{1-p} (\log x)^{1-p} |_2^N & \text{if } p \neq 1 \\
\log(\log x) |_2^N & \text{if } p = 1
\end{cases}
$$

Note that for $p > 1$, $\lim_{N \to \infty} \int_2^N f(x)dx = \frac{1}{p-1} (\log 2)^{1-p}$, so it converges.

For $p \geq 1$, obviously that the integral converges to infinite, so the series diverges to infinite as well. \hfill \Box