Problem. 17.5

Proof. (a) Let’s prove by induction. It is obviously true for \( n = 1 \), i.e., \( f(x) = x \) is continuous. Suppose \( x^n \) is continuous, then \( g(x) = x^{n+1} = x^n \cdot x \) is a multiplication of two continuous functions, so it is also continuous. That is to say, \( x^n \) is continuous for all \( n \in \mathbb{N} \).

(b) Every polynomial is a linear combination of continuous functions \( x, \ldots, x^n \) (by (a)), so it is also continuous. □

Problem. 17.7 (b)

Proof. For any \( x > 0 \), \( |x| = x \), so it is continuous at any points \( x > 0 \). For any \( x < 0 \), \( |x| = -x \), so it is also continuous at any points \( x < 0 \). We suffice to show it is continuous at \( x = 0 \). For any \( \epsilon > 0 \), pick \( \delta = \epsilon \), then for any \( x \) s.t. \( |x| < \delta \), we have \( |x| < \delta = \epsilon \), which means \( |x| \) is continuous at \( x = 0 \). Overall, \( |x| \) is continuous on \( \mathbb{R} \). □

Problem. 17.8

Proof. (a) For any \( x \) in the domain of \( f \) and \( g \), WLOG, say \( f(x) \leq g(x) \), then we have

\[
RHS = \frac{1}{2}(f(x) + g(x)) - \frac{1}{2}|f(x) - g(x)|
\]

\[
= \frac{1}{2}(f(x) + g(x)) - \frac{1}{2}[g(x) - f(x)]
\]

\[
= f(x)
\]

\[
= \min(f, g)(x) = LHS
\]

(b) Again, WLOG, suppose that \( f(x) \leq g(x) \), then \( \min(f, g)(x) = f(x) \) and \( -\max(-f, -g)(x) = f(x) \) (because \( -f(x) \geq -g(x) \)).

(c) We know that composition of continuous functions is continuous and \( |x| \) is continuous, so \( |h| \) is continuous for \( h \) is continuous. By part (a), we know that \( \min(f, g) \) is a summation of continuous, so it is also continuous. □

Problem. 17.9 (b)

Proof. Given \( \epsilon > 0 \), let \( \delta = \epsilon^2 \), then for any \( 0 < x < \delta \), we have \( |f(x) - 0| = \sqrt{x} < \sqrt{\delta} = \epsilon \). □

Problem. 17.10 (b)

Proof. Consider the sequence \( x_n = \frac{1}{2^{n+\pi} \pi / 2} \), then \( x_n \to 0 \) and \( g(x_n) = \sin(x_n) = 1 \) for all \( n \). So \( \lim g(x_n) = 1 \), which is not equal to \( g(0) = 0 \), which means it is not continuous at \( x_0 = 0 \). □
Problem. 18.5 (a)

Proof. Consider the function \( h(x) = f(x) - g(x) \) on the domain (obviously it is continuous), note that \( h(a) = f(a) - g(a) \geq 0 \) and \( h(b) = f(b) - g(b) \leq 0 \). If one of \( h(a) \) and \( h(b) \) equal to 0, then \( f(a) = h(a) \) (or \( f(b) = g(b) \)). If neither \( h(a) = 0 \) nor \( h(b) = 0 \), i.e., \( h(a) > 0 \) and \( h(b) < 0 \), then by the Intermediate Value Theorem, there exists \( x_0 \in (a, b) \) s.t. \( h(x_0) = 0 \), which means \( f(x_0) = g(x_0) \). So overall, there is one \( x_0 \in [a, b] \) s.t. \( f(x_0) = g(x_0) \). \(\square\)

Problem. 18.6

Proof. Consider the function \( f(x) = x - \cos x \) (obviously it is continuous), \( f(0) = -1 < 0 \) and \( f(\pi/2) = \pi/2 > 0 \). Then by the Intermediate Value Theorem, there exists \( x_0 \in (0, \pi/2) \) s.t. \( f(x_0) = x - \cos x \), i.e., \( x_0 = \cos x_0 \). \(\square\)

Problem. 18.8

Proof. Because \( f(a)f(b) < 0 \), so WLOG, \( f(a) < 0 \) and \( f(b) > 0 \), then by the Intermediate Value Theorem, there exists \( x_0 \) between s.t. \( f(x_0) = 0 \). \(\square\)

Problem. 18.10

Proof. Define a continuous function \( g(x) = f(x+1) - f(x) \) on \([0, 1]\). Note that \( g(0) = f(1) - f(0) = f(1) - f(2) = -g(1) \). If \( g(0) = 0 \), then we can just let \( x = 0 \) and \( y = 1 \), so \(|y - x| = 1\) and \( f(1) = f(0) \). If \( g(0) \neq 0 \), then \( g(0)g(1) < 0 \), then by problem 18.8, there exists \( x_0 \in (0, 1) \) between s.t. \( g(x_0) = 0 \). So just let \( x = x_0 \) and \( y = x_0 + 1 \), we have \(|y - x| = 1\) and \( f(x) = f(y) \). \(\square\)