Problem 19.1.

Part b. Since $f$ is continuous on $[0, 1]$, by theorem 19.2, $f$ is uniform continuous on $[0, 1]$.

Part d. $f$ is not uniform continuous on $\mathbb{R}$. Take $\epsilon = 1$, for any given $\delta$, take $x = \frac{1}{\sqrt{\delta}}$ and $y = x + \delta/3$. Then:

$$|f(y) - f(x)| = |(x + \delta/3)^3 - x^3| = |\delta x^2 + \frac{1}{3} \delta^2 x + \left(\frac{\delta}{3}\right)^3|$$

$$> |\delta x^2| = 1$$

That contradicts with the definition of uniform continuous. So $f$ is not uniform continuous on $\mathbb{R}$.

Problem 19.2 (b).

Proof. For any given $\epsilon$, we take $\delta = \min\{\frac{\epsilon}{7}, 1\}$. Suppose $y = x + a$ where $|a| < \delta$. Then since $|x| \leq 3$:

$$|f(y) - f(x)| = |(x + a)^2 - x^2| = |2ax + a^2| \leq |2ax| + |a^2|$$

$$\leq 6|a| + |a^2| \leq 7|a| \leq 7\delta \leq \epsilon$$

Problem 19.3 (a).
Proof. For any given \(\epsilon\), we take \(\delta = 4\epsilon\). Suppose \(y = x + a\) where \(|a| < \delta\). Then since \(x \geq 1\) and \(y \geq 1\):

\[
|f(y) - f(x)| = |(1 - \frac{1}{y+1}) - (1 - \frac{1}{x+1})| = \frac{a}{(x+1)(y+1)}
\]

\[
\leq \frac{a}{4} \leq \delta/4 = \epsilon
\]

\[\square\]

Problem 19.6.

Part a.

\[
f'(x) = (x^{1/2})' = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}
\]

Notice that when \(x \to 0\), \(\frac{1}{\sqrt{x}} \to \infty\). Thus \(f'(x)\) is not bounded on \((0, 1]\).

Since \(f(x)\) is defined and continuous on \([0, 1]\), by theorem 19.2, \(f\) is uniform continuous on \([0, 1]\) and thus uniform continuous on \((0, 1]\).

This tell us that the converse of theorem 19.6 is not true. Having the derivative unbounded does not mean that the function is not uniform continuous.

Part b. \(|f'(x)| = |\frac{1}{2\sqrt{x}}| < \frac{1}{2}\) when \(x \in [1, \infty)\). So the derivative is bounded on \([1, \infty)\). Thus by theorem 19.6, \(f\) is uniform continuous on \([1, \infty)\).

Problem 19.8.

Part a. We have \(f'(x) = \cos x\) so that \(|f'(x)| \leq 1\). Given \(x < y\), by the mean value theorem, there exists \(x < a < y\) such that \(\frac{\sin x - \sin y}{x - y} = \cos a\). So we have that:

\[
\frac{|\sin x - \sin y|}{|x - y|} = |\cos a| \leq 1
\]

Move the \(|x - y|\) to the right side of the inequality we have that \(|\sin x - \sin y| \leq |x - y|\).

Part b. For any given \(\epsilon\), we take \(\delta = \epsilon\). The for \(|x - y| < \delta\):

\[
|\sin x - \sin y| \leq |x - y| < \delta = \epsilon
\]

So \(\sin x\) is uniform continuous on \(\mathbb{R}\).