Math 142A Introduction to Analysis

Name (PRINT):_____________________________  PID:____________

Signature: ________________________________

**Important Instructions:** No books, notes, cell phones, or any other electronic devices may be used during the exam. Do not start the exam until instructed to do so. You cannot use any result in your homework unless you reprove it.

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(5) 1 (a). State the rational zero theorem.

Suppose $c_0, \ldots, c_n \in \mathbb{Z}$ for $c_0 \neq 0 \neq c_n$ and $\frac{p}{q} \in \mathbb{Q}$, $\gcd(p, q) = 1$.

is a solution of \( c_n x^n + \cdots + c_1 x + c_0 = 0 \).

Then \( p \mid c_0 \), \( q \mid c_n \).

(10) 1 (b). Use the above theorem to prove that $\sqrt{2 + \sqrt{3}}$ is NOT a rational number.

**Proof:** Suppose $\sqrt{2 + \sqrt{3}} = \frac{p}{q} \in \mathbb{Q}$ s.t. $\gcd(p, q) = 1$.

Then $2 + \sqrt{3} = (\frac{p}{q})^2 \Rightarrow \left[(\frac{p}{q})^2 - 2\right]^2 = 3$

$$((\frac{p}{q})^4 - 4(\frac{p}{q})^2 + 1 = 0$$

Then by the RZT, $p \mid 1$ and $q \mid 1$.

$\Rightarrow \frac{p}{q} = \pm 1$. Note that neither 1 or -1 is a solution of \( x^4 - 4x^2 + 1 = 0 \).

So it does not have rational solution.

So $\frac{p}{q}$ does not exist. $\sqrt{2 + \sqrt{3}} \notin \mathbb{Q}$
2. Let \( \{a_n\} \) and \( \{b_n\} \) be sequences such that \( a_n \to a, b_n \to b \) and \( c_n \to c \). Use the definition of limit to prove that \( a_n + 2b_n + c_n \to a + 2b + c \). To use the definition means you must use the \( \varepsilon, N \) type argument.

Given \( \varepsilon > 0 \), \( \exists N_1, N_2, N_3 \in \mathbb{N} \) s.t. for all \( \forall n > N_x \Rightarrow |a_n - a| < \frac{\varepsilon}{4} \) \( \forall n > N_2 \Rightarrow |b_n - b| < \frac{\varepsilon}{4} \) \( \forall n > N_3 \Rightarrow |c_n - c| < \frac{\varepsilon}{4} \).

Let \( N = \max\{N_1, N_2, N_3\} \).

Then for \( \forall n > N \), we have

\[
|a_n + 2b_n + c_n - (a + 2b + c)| = |(a_n - a) + 2(b_n - b) + (c_n - c)|
\]

\[
\leq |a_n - a| + 2|b_n - b| + |c_n - c|
\]

\[
< \frac{\varepsilon}{4} + 2 \cdot \frac{\varepsilon}{4} + \frac{\varepsilon}{4}
\]

\[
= \varepsilon.
\]

So by definition, \( a_n + 2b_n + c_n \to a + 2b + c \).

(5) 3. State the monotone convergence theorem.

A bounded monotonic sequence converges.
4. Let $a_n$ be a sequence be given by
\[ a_1 = 2, \quad a_{n+1} = \frac{a_n}{2} + \frac{1}{2a_n}. \]

(a). Prove by induction that $a_n > 1$ for all $n \geq 1$.

\[ a_1 = 2 > 1 \]

Suppose $a_n > 1$.

Then $a_{n+1} = \frac{a_n}{2} + \frac{1}{2a_n}$

\[ a_{n+1} - 1 = \frac{a_n^2 + 1}{2a_n} = \frac{(a_n - 1)^2}{2a_n} > 0 \]

$\Rightarrow a_{n+1} > 1$.

So it is proved by induction.

(b). Prove the sequence is decreasing, i.e., $a_{n+1} \leq a_n$.

Suppose that $n > 1$.

\[ a_{n+1} = \frac{a_n}{2} + \frac{1}{2a_n} \leq \frac{a_n}{2} + \frac{1}{2a_n} \leq \frac{a_n}{2} + \frac{a_n}{2} = a_n \, \checkmark \]

(c). Prove that $a_n$ converges and find its limit.

By the Monotone Convergent Theorem, b/c it is decreasing and bounded below. So it converges.

\[ \lim a_{n+1} = \lim \frac{a_n}{2} + \lim \frac{1}{2a_n} \]

\[ a = \frac{a}{2} + \frac{1}{2a} \Rightarrow a^2 - a = 0 \Rightarrow a = 0, \pm 1 \]

$\Rightarrow a = 1$ (b/c $a_n > 1$)
5. Compute \( \lim_{n \to \infty} (\sqrt{4n^2 + n} - 2n) \). You can use any theorem we taught in the book, not necessarily the definition of limit.

**Proof:** Note that for any \( n \),

\[
\sqrt{4n^2 + n} - 2n = \left( \frac{\sqrt{4n^2 + n} - 2n}{1} \right) \frac{\sqrt{4n^2 + n} + 2n}{\sqrt{4n^2 + n} + 2n}
\]

\[
= \frac{4n^2 + n - 4n^2}{\sqrt{4n^2 + n} + 2n} = \frac{1}{\sqrt{4n^2 + n} + 2n} \approx \frac{1}{4 + \frac{1}{n} + 2}
\]

So

\[
\lim_{n \to \infty} (\sqrt{4n^2 + n} - 2n) = \lim_{n \to \infty} \frac{1}{\sqrt{4 + \frac{1}{n} + 2}} = \frac{1}{\sqrt{4 + 2}} = \frac{1}{4}
\]
(3) 7. Let $A, B$ be two subsets of $\mathbb{R}$. We define $A + B$ as the following:

$$A + B = \{ a + b : a \in A, b \in B \}.$$

(a). Let $S = [1, 5], T = \{ 2, 3 \}$. Find $S + T$.

Claim: $S + T = [3, 8]$  

For any $s + t \in S + T$, $\implies 1 \leq s \leq 5 \implies t = 2$ or $3$  

$\implies 3 \leq s + t \leq 8 \implies S + T \subseteq [3, 8]$.

For $x \in [3, 8]$, if $x \in [3, 7]$, $x = s + 2$ for $s \in [1, 5]$.

if $x \in [4, 8]$, $x = s + 3$ for $s \in [1, 5]$.

So overall, $S + T = [3, 8]$.

(b). Prove for any two nonempty sets $A, B$, it holds that $\text{sup}(A + B) = \text{sup}A + \text{sup}B$.

At first suppose $A \neq B$ are both bounded above.

For all $x \in A + B$, $x = a + b$ for $a \in A, b \in B$.

$\implies x = a + b \leq \text{sup}A + \text{sup}B$. $\implies \text{sup}A + \text{sup}B$ is an upper bound of $A + B$.

$\implies \text{sup}(A + B) \leq \text{sup}A + \text{sup}B$.

On the other hand, given $\varepsilon > 0$. $\exists a \in A, b \in B$ s.t.  

\[ \text{sup}A \leq a + \varepsilon/2, \quad \text{sup}B \leq b + \varepsilon/2. \]

$\implies \text{sup}A + \text{sup}B \leq a + b + \varepsilon \leq \text{sup}(A + B) + \varepsilon$ (for $\forall \varepsilon > 0$)

$\implies \text{sup}A + \text{sup}B \leq \text{sup}(A + B)$

Overall, $\text{sup}A + \text{sup}B = \text{sup}(A + B)$.

If either $A$ or $B$ unbounded above, then both sides equal $\infty$. 

\[ \varepsilon \]