**Important Instructions:** No books, notes, cell phones, or any other electronic devices may be used during the exam. Do not start the exam until instructed to do so. You cannot use any result in your homework unless you reprove it.
1. Let \( a_n = e^{-n} + \cos(n^2 + 10) + \sin(2n) + \frac{n}{n^2 + 1} \) for \( n \geq 1 \). Prove \((a_n)\) has a convergent subsequence.

\[
|a_n| \leq |e^{-n}| + |\cos(n^2 + 10)| + |\sin(2n)| + \left| \frac{n}{n^2 + 1} \right| \\
\leq 1 + 1 + 1 + 1 = 4
\]

so \((a_n)\) is bounded. By the Bolzano-Weierstrass Theorem, it has a convergent subseq.

2 (a). State the definition of a Cauchy sequence.

\[
\{s_n\} \text{ Cauchy } \iff \forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n, m \geq N, \text{ we have } |s_n - s_m| < \varepsilon
\]

2 (b). Use the definitions to prove that a convergent sequence must be Cauchy.

Suppose \( s_n \to s \)

\[\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n, m \geq N, |s_n - s| < \frac{\varepsilon}{2} \]

\[|s_m - s| < \frac{\varepsilon}{2} \]

So \( |s_n - s_m| = |s_n - s + s - s_m| \leq |s_n - s| + |s_m - s| \)

\[< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \]

so \( \{s_n\} \) is Cauchy.
3. Let \((s_n)\) be a sequence such that \(|s_{n+1} - s_n| < \frac{1}{n^{2+1}}\) for all \(n \geq 1\). Prove that \((s_n)\) is convergent and thus Cauchy.

Define a sequence \(a_n = s_{n+1} - s_n\).

Note that \(\sum_{k=1}^{n} a_k = s_{n+1} - s_n + s_n - s_{n-1} + \cdots + s_2 - s_1 = s_{n+1} - s_1\).

On the other hand, \(|a_n| < \frac{1}{n^{2+1}}\), because \(\sum_{n=1}^{\infty} \frac{1}{n^{2+1}} < \infty\), by the comparison test. \(\sum_{k=1}^{n} a_k\) converges (absolutely).

\(\Rightarrow\) \(\{s_{n+1} - s_1\}\) converges.

\(\Rightarrow\) \(\{s_n\}\) converges (by Cauchy).
4. Determine whether the following statements are true or false. Just write T (for True) or F (for False) in the parentheses.

(1). There exists a sequence \((a_n)\) such that \(\sum_{n=1}^{\infty} a_n\) converges and \(\sum_{n=1}^{\infty} a_n^2\) diverges. \(\text{( } T \text{ )}\).

(2). There exists a sequence \((a_n)\) such that \(\sum_{n=1}^{\infty} a_n\) diverges and \(\sum_{n=1}^{\infty} a_n^2\) converges. \(\text{( } T \text{ )}\). \(\sum \frac{1}{n} = \infty\) but \(\sum \frac{1}{n^2} < \infty\).

(3). The series \(\sum_{n=1}^{\infty} \frac{n}{\sqrt{n!}}\) diverges. \(\text{( } T \text{ )}. \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{n+1}{n} \int_{n+1}^{\infty} \frac{1}{t} dt = 0 < 1\).

(4). The series \(\sum_{n=1}^{\infty} \frac{2+\cos n}{n^2}\) converges. \(\text{( } T \text{ )}. \lim_{n \to \infty} \frac{\cos n}{n^2} < \lim_{n \to \infty} \frac{2+1}{n^2} = \frac{3}{n^2}\).

(5). There exists a sequence \(a_n\) such that \(\sum_{n=1}^{\infty} |a_n|\) converges and \(\sum_{n=1}^{\infty} a_n\) diverges. \(\text{( } F \text{ )}\). Converges absolutely implies convergent.

5 (a). Let \(s_n = (-1)^n(\sqrt{n+1} - \sqrt{n})\) for \(n \geq 1\). Does \(\sum_{n=1}^{\infty} s_n\) converge? Justify your answer.

Note that \(s_n = (-1)^n \left( \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \right) \left( \sqrt{n+1} + \sqrt{n} \right)\)

\[= (-1)^n \frac{1}{\sqrt{n+1} + \sqrt{n}}\]

Note \(\frac{1}{\sqrt{n+1} + \sqrt{n}}\) is decreasing and goes to zero.

By the alternating series test, \(\{ s_n \}\) converges.
(3) 5 (b). Is the following statement true or false? If true, give a formal proof. If false, give a counterexample.

If all $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \frac{a_n}{a_n+1}$ converges.

True. Claim (08) $\frac{a_n}{a_n+1} \leq a_n$

If $a_n = 0$, obviously.

If $a_n > 0$, LHS = $\frac{1}{1 + \frac{1}{a_n}} \leq \frac{1}{a_n} = a_n$.

So by the comparison test, $\sum_{n=1}^{\infty} \frac{a_n}{a_n+1} < \infty$.

(3) 5 (c). Is the following statement true or false? If true, give a formal proof. If false, give a counterexample.

If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \frac{a_n+2}{a_n+1}$ diverges.

True. Because $\sum_{n=1}^{\infty} a_n$ converges, $\Rightarrow a_n \rightarrow 0$.

So $\frac{a_n+2}{a_n+1} \xrightarrow{n \rightarrow \infty} \frac{0+2}{0+1} = 2 \neq 0$.

So $\left( \sum_{n=1}^{N} \frac{a_n+2}{a_n+1} \right)$ diverges.
The following question is optional and you get bonus points if you solve it.

(2) 6(a). Do you have any suggestions to improve the quality of future classes?

Thank you all for your suggestions!!

(1) 6(b). Does \( \sum_{n=2}^{\infty} \frac{1}{\sqrt{n \log n}} \) converge? Justify your answer.

No. Note that \( \log n < n^{\frac{1}{2}} \) for all large \( n \).

\[ f(x) = \log x - x^{\frac{1}{2}} \quad \text{for} \quad x \geq 1 \quad \Rightarrow \quad f'(x) = \frac{1}{x} - \frac{1}{2} x^{-\frac{1}{2}} < 0 \quad \text{for} \quad x > 4 \]

Also \( f(4) = \log 4 - 4^{\frac{1}{2}} = \log(2^2) - 2 < \log e^2 - 2 = 2 - 2 = 0 \)

So \( f(n) = \log n - n^{\frac{1}{2}} < 0 \)

So \( \frac{1}{\sqrt{n \log n}} > \frac{1}{\sqrt{n \cdot n^{\frac{1}{2}}}} = \frac{1}{n} \), where \( \frac{1}{n} \to 0 \).

By the comparison test, so \( \sum_{n=2}^{\infty} \frac{1}{\sqrt{n \log n}} = \infty \).