§ 29 The mean value theorem

Thm 29.1
Let \( f \) be defined on an open interval containing \( x_0 \).
If \( f \) assumes its maximum or minimum at \( x_0 \), and if
\( f \) is differentiable at \( x_0 \), then \( f'(x_0) = 0 \)

Pf: Suppose \( f \) is defined on \((a, b)\) where \( a < x_0 < b \).
we have two cases.

Case I: Suppose \( f \) assumes its maximum at \( x_0 \).
we need to prove \( f'(x_0) = 0 \).

Recall \( f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \)

\[ = \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \]

\[ = \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} \]

when \( x \to x_0^+ \), \( f(x) \leq f(x_0) \), and \( x > x_0 \)
\[ \Rightarrow \frac{f(x) - f(x_0)}{x - x_0} \leq 0 \Rightarrow \lim_{x \to x_0^+} \frac{f(x) - f(x_0)}{x - x_0} \leq 0 \]
\[ \Rightarrow f'(x_0) \leq 0 \]

when \( x \to x_0^- \), \( f(x) \leq f(x_0) \), and \( x < x_0 \)
\[ \Rightarrow \frac{f(x) - f(x_0)}{x - x_0} > 0 \Rightarrow \lim_{x \to x_0^-} \frac{f(x) - f(x_0)}{x - x_0} > 0 \]
\[ \Rightarrow \frac{f(x) - f(x_0)}{x - x_0} \geq 0 \Rightarrow \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \geq 0 \]

\[ \Rightarrow f'(x_0) \geq 0 \]

Therefore, \( f'(x_0) = 0 \)

**Case II.** Suppose \( f \) assume its minimum at \( x_0 \).

Then \( -f \) assume its maximum at \( x_0 \).

By Case I, \(( -f)'(x_0) = 0 \Rightarrow f'(x_0) = 0 \).

Recall we proved the following in Chapter 3:

A continuous function on a bounded closed interval \([a,b]\) achieves its maximum and minimum. That is, \( \exists x_0, y_0 \in [a,b], \) s.t. \( f(x_0) \leq f(x) \leq f(y_0), \forall x \in [a,b]. \) \( \Leftarrow \) Thm 18.1.

**Thm 29.2 (Rolle's Thm)**

Let \( f \) be continuous on \([a,b]\) and differentiable on \((a,b)\). Assume \( f(a) = f(b) \). Then \( \exists \) a least one \( x_0 \in (a,b) \) s.t. \( f'(x_0) = 0 \)

**Pf:** By Thm 18.1, \( \exists x_0, y_0 \in [a,b] \) s.t. \( f(x_0) \leq f(x) \leq f(y_0) \) for all \( x \in [a,b] \). If \( x_0, y_0 \) are both endpts of \([a,b]\), then \( f \) is a constant function as \( f(a) = f(b) \). This implies \( f'(x) = 0 \) for all \( x \in (a,b) \). Otherwise, at least one of \( x_0, y_0 \)
\[ f'(x) \to \text{ for all } x \in (a, b). \text{ Otherwise, at least one of } x_0, y_0 \text{ is not an endpoint. } \Rightarrow f \text{ assumes either a maximum or a minimum at an interior point } x \in (a, b). \text{ By Thm 29.1, } f'(x) = 0. \]

**Thm 29.3. (Mean Value Thm)**

Let \( f \) be a continuous function on \([a, b]\) that is differentiable on \((a, b)\). Then \( \exists \ x \in (a, b) \) s.t. \( f'(x) = \frac{f(b) - f(a)}{b - a} \).

**Pf:** Let \( L \) be the function whose graph is the straight line connecting \((a, f(a))\) and \((b, f(b))\) (see the picture), i.e. \( y = L(x) = \frac{f(b) - f(a)}{b - a} (x - a) + f(a) \).

In particular, \( L(a) = f(a) \), \( L(b) = f(b) \), \( L'(x) = \frac{f(b) - f(a)}{b - a} \).

Set \( g(x) = f(x) - L(x) \). Then \( g \) is continuous on \([a, b]\) and is differentiable on \((a, b)\).

Moreover, \( g(a) = f(a) - L(a) = 0 \), \( g(b) = f(b) - L(b) = 0 \).

By Rolle's Theorem, \( \exists \ x_0 \in (a, b), \) s.t. \( g'(x_0) = 0 \).

But \( g'(x_0) = f'(x_0) - L'(x_0) \)

\[ \Rightarrow f'(x_0) = \frac{f(b) - f(a)}{b - a}. \]
But \( g'(x_0) = f'(x_0) - L'(x_0) \)

\[ f'(x_0) = \frac{f(b) - f(a)}{b - a} \]

**Corollary 29.4.** Let \( f \) be a differentiable function on \((a, b)\) s.t \( f'(x) = 0 \) for \( \forall x \in (a, b) \). Then \( f \) is a constant function on \((a, b)\).

**pf.** If \( f \) is not constant on \((a, b)\), then \( \exists x_1, x_2 \) s.t \( a < x_1 < x_2 < b \) and \( f(x_1) \neq f(x_2) \).

By the Mean Value Thm, \( \exists x_0 \in (x_1, x_2) \) s.t

\[ f'(x_0) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \neq 0. \]

This is a contradiction.

**Corollary 29.5.** Let \( f, g \) be differentiable on \((a, b)\) s.t \( f' = g' \) on \((a, b)\). Then \( \exists a \) constant \( c \) s.t

\[ f(x) = g(x) + c \]
\[ f' = g' \text{ on } (a, b). \text{ Then } \exists \text{ a constant } c \text{ s.t.} \]
\[ f(x) = g(x) + c \text{ for all } x \in (a, b) \]

\textbf{Pf.: Exercise. Hint: Apply Corollary 29.4 to } f-g. \]

\textbf{Remark: Recall the following fact from Calculus:}

If \( F_1, F_2 \) are both anti-derivative of \( f \) on \( (a,b) \),
then \( F_1 = F_2 + C \).
prove the above fact.

\textbf{Defn 29.6}

Let \( f \) be a function on an interval \( I \). We say

1. \( f \) is strictly increasing on \( I \) if
   \[ x_1, x_2 \in I \text{ and } x_1 < x_2 \Rightarrow f(x_1) < f(x_2) \]
2. \( f \) is strictly decreasing on \( I \) if
   \[ x_1, x_2 \in I \text{ and } x_1 < x_2 \Rightarrow f(x_1) > f(x_2) \]
3. \( f \) is increasing on \( I \) if
   \[ x_1, x_2 \in I \text{ and } x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2) \]
4. \( f \) is decreasing on \( I \) if
   \[ x_1, x_2 \in I \text{ and } x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2) \]
Corollary 29.7

Let $f$ be a differentiable function on $(a, b)$. Then

(i) If $f'(x) > 0$ on $(a, b) \Rightarrow f$ is strictly $\uparrow$

(ii) If $f'(x) < 0$ on $(a, b) \Rightarrow f$ is strictly $\downarrow$

(iii) If $f'(x) \geq 0$ on $(a, b) \Rightarrow f$ is $\uparrow$

(iv) If $f'(x) \leq 0$ on $(a, b) \Rightarrow f$ is $\downarrow$

Proof: (i) Consider $x_1, x_2 \in (a, b)$ with $x_1 < x_2$.

By M.V.T., $\exists \xi \in (x_1, x_2)$ s.t

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\xi) > 0$$

$\Rightarrow f(x_2) > f(x_1) \Rightarrow f$ is strictly $\uparrow$.

(ii), (iii), (iv): Exercise.

Q: Let $f$ be differentiable on $(a, b)$

True or False

$f$ strictly $\uparrow$ on $(a, b) \Rightarrow f'(x) > 0$ for all $x \in (a, b)$.

A: False.

Thm 29.8 (Intermediate Value Thm for Derivatives)

Let $f$ be a differentiable function on $(a, b)$. If $a < x_1 < x_2 < b$,
Let \( f \) be a differentiable function on \((a, b)\). If \( a < x_1 < x_2 < b\), \( c \) lies between \( f'(x_1) \) and \( f'(x_2) \), then \( \exists \ x \in (x_1, x_2) \) s.t. \( f'(c) = 0 \).

**Pf:** We may assume \( f'(x) < c < f'(x_2) \). Let \( g(x) = f(x) - cx \) for \( x \in (a, b) \). Then \( g'(x_1) < 0 < g'(x_2) \).

Since \( g \) is continuous on \([x_1, x_2]\), \( \Rightarrow g \) achieves its minimum in \([x_1, x_2]\), i.e., \( \exists \ x_0 \in [x_1, x_2], s.t \ g(x_0) \leq g(x), \forall x \in [x_1, x_2] \).

**Claim 1:** \( x_0 \neq x_1 \)

**Pf of claim 1:** Note \( g'(x_1) = \lim_{y \to x_1} \frac{g(y) - g(x_1)}{y - x_1} < 0 \).

\( \Rightarrow \frac{g(y) - g(x_1)}{y - x_1} < 0 \) for \( y \) close to \( x_1 \) and \( y > x_1 \)

\( \Rightarrow g(y) < g(x_1) \) for \( y \) close to \( x_1 \) and \( y > x_1 \)

\( \Rightarrow g(x_1) \) is not the minimum of \( g \) on \([x_1, x_2]\).

**Claim 2:** \( x_0 \neq x_2 \)

**Pf of claim 2:** Note \( g'(x_2) = \lim_{y \to x_2} \frac{g(y) - g(x_2)}{y - x_2} > 0 \).

\( \Rightarrow \frac{g(y) - g(x_2)}{y - x_2} > 0 \) for \( y \) close to \( x_2 \) and \( y < x_2 \)

\( \Rightarrow g(y) < g(x_2) \) for \( y \) close to \( x_2 \) and \( y < x_2 \)
- \( g(y) < g(x) \) for \( y \) close to \( x \) when \( y \to 0^+ \)

\[ \Rightarrow g(x_2) \text{ cannot be the minimum of } g \text{ on } [x_1, x_2]. \]

Hence \( x_0 \in (x_1, x_2) \). Thus by Thm 29.1, \( g'(x_0) = 0 \)

This implies \( f'(x_0) = 0 \).

**Thm 29.9**

Let \( f \) be a one-to-one continuous function on an open interval \( I \), and \( J = f(I) \). If \( f \) is differentiable at \( x_0 \in I \) and if \( f(x_0) \neq 0 \), then \( f^{-1} \) is differentiable at \( y_0 = f(x_0) \) and

\[
(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f(f^{-1}(y_0))}
\]

**Pf:** P237 in the book.

**Ex.**

Let \( f(x) = \sin x \) on \([-\frac{\pi}{2}, \frac{\pi}{2}]\).

Set \( g(y) = f^{-1}(y) = \sin^{-1}(y) = \arcsin(y) \)

For any \( x_0 \in (-\frac{\pi}{2}, \frac{\pi}{2}) \), let \( y_0 = f(x_0) \in (-1, 1) \)

Note \( f'(x_0) = \cos x_0 \neq 0 \).

By Thm 29.9,

\[
g'(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{\cos(\arcsin(y_0))}
\]
By the following graph

\[ \frac{1}{\sqrt{1-y^2}} \]

Let \( x_0 = \arcsin y_0 \)

\[ \Rightarrow \cos(\arcsin y_0) = \cos x_0 = \sqrt{1-y_0^2} \]

\[ \Rightarrow g'(y_0) = \sqrt{1-y_0^2}, \quad y_0 \in (-1, 1) \]

3 L'Hospital's Rule

Thm 30.2 (L'Hospital's Rule)

Let \( s \) signify \( a, a^+, a^-, \infty \) or \( -\infty \) where \( a \in \mathbb{R} \), and

Let \( f, g \) be differentiable functions. Assume

\[ \lim_{x \to s} \frac{f'(x)}{g'(x)} = L \]

If \( \lim_{x \to s} f(x) = \lim_{x \to s} g(x) = 0 \).

or

\[ \lim_{x \to s} |g(x)| = +\infty \]

then

\[ \lim_{x \to s} \frac{f(x)}{g(x)} = L. \]