33 properties of the Riemann Integral

Thm 33.1 Every monotonic function \( f \) on \([a, b]\) is integrable.

\[ f(a) < f(b), \quad \text{if } f(a) = f(b), \text{ then } f \text{ is constant on } [a, b], \text{ which is clearly integrable. Now assume } f(a) < f(b). \]

Note \( f \) is bdd as \( f(a) \leq f(x) \leq f(b) \) for \( x \in [a, b] \).

Take any partition \( P = \{a = t_0 < t_1 < \cdots < t_n = b\} \) with mesh less than \( \frac{\varepsilon}{f(b) - f(a)} \). Then

\[
U(f, P) - L(f, P) = \sum_{k=1}^{n} \left( M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) \right) (t_k - t_{k-1})
\]

\[ = \sum_{k=1}^{n} [f(t_k) - f(t_{k-1})] (t_k - t_{k-1}) \]

Since \( \text{mesh}(P) < \frac{\varepsilon}{f(b) - f(a)} \), we have \( t_k - t_{k-1} < \frac{\varepsilon}{f(b) - f(a)} \), \( \forall 1 \leq k \leq n. \)

\[ \Rightarrow \quad U(f, P) - L(f, P) < \sum_{k=1}^{n} [f(t_k) - f(t_{k-1})] \frac{\varepsilon}{f(b) - f(a)} \]

\[ = [f(b) - f(a)] \frac{\varepsilon}{f(b) - f(a)} = \varepsilon. \]

Case II. Suppose \( f \) is decreasing on \([a, b]\). Ex.

Thm 33.2 Every continuous function \( f \) on \([a, b]\) is integrable.

\[ \text{Pf: Let } \varepsilon > 0. \quad \text{Recall } f \text{ continuous on } [a, b] \Rightarrow f \text{ is} \]
Pf. Let $\varepsilon > 0$. Recall $f$ is continuous on $[a, b] \Rightarrow f$ is uniformly continuous on $[a, b]$, $\Rightarrow \exists \delta > 0$ s.t.

\[ \forall x, y \in [a, b] \text{ and } |x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{b - a} \quad (1) \]

Consider any partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ where

\[ \max \{t_k - t_{k-1} : k = 1, 2, \ldots, n\} < \delta. \]

Since $f$ attains its max and min on each $[t_{k-1}, t_k]$,

it follows from (1) that

\[ M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) < \frac{\varepsilon}{b - a} \]

for every $k$. $\Rightarrow$

\[ U(f, P) - L(f, P) = \sum_{k=1}^{n} (M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]))(t_k - t_{k-1}) \]

\[ < \sum_{k=1}^{n} \frac{\varepsilon}{b - a} (t_k - t_{k-1}) = \varepsilon. \]

By Thm 32.5, $f$ is integrable.

Thm 33.3 Let $f, g$ be integrable functions on $[a, b]$, and let $c$ be a real number. Then

(i) $cf$ is integrable and $\int_{a}^{b} cf = c \int_{a}^{b} f$

(ii) $f + g$ is integrable and $\int_{a}^{b} (f + g) = \int_{a}^{b} f + \int_{a}^{b} g$.

Pf. Read P.282 - 283.

Thm 33.4 (i) If $f$ and $g$ are integrable on $[a, b]$ and if

\[ f(x) \leq g(x) \text{ for } x \in [a, b], \text{ then } \int_{a}^{b} f \leq \int_{a}^{b} g. \]

(ii) If $g$ is a continuous nonnegative function
If \( g \) is a continuous nonnegative function on \([a, b]\) and if \( \int_a^b g = 0 \), then \( g \) is identically 0 on \([a, b]\).

\[ \text{Pf (i):} \quad \text{Let } h = g - f. \text{ By Thm 33.3, } h \text{ is integrable on } [a, b]. \]

Since \( h(x) \geq 0 \) on \([a, b]\), then it is clear that \( L(h, P) \geq 0 \) for all partition \( P \) of \([a, b]\), so
\[ \int_a^b h = L(h) \geq 0. \]

Apply Thm 33.3 again,
\[ \int_a^b h = \int_a^b g + \int_a^b (-f) = \int_a^b g - \int_a^b f \geq 0, \]
\[ \Rightarrow \int_a^b g \geq \int_a^b f. \]

(ii). Suppose \( g \) is not identically 0. Then \( \exists \ x_0 \in [a, b] \), such that \( f(x_0) > 0 \). By the continuity of \( f \), \( \exists \) an interval \([c, d] \subseteq [a, b] \) containing \( x_0 \), such that \( g(x) > \frac{f(x_0)}{2} \) on \([c, d] \). Then
\[ \int_c^d g \geq \int_c^d g \geq \frac{f(x_0)}{2} (d - c) > 0. \]

This contradicts with \( \int_a^b g = 0. \)

Remark: Let \( f, g \) be continuous on \([a, b]\), and \( f(x) \leq g(x) \) on \([a, b]\). Assume \( g - f \) is not identically zero. Then
\[ \int_a^b f < \int_a^b g. \]
Thm 33.5

If $f$ is integrable on $[a,b]$, then $|f|$ is integrable on $[a,b]$ and $\int_a^b |f| \leq \int_a^b |f|.$

**Proof.** Step 1: prove $|f|$ is integrable

**Claim:** For any subset $S$ of $[a,b]$, we have

$$M(|f|, S) - m(|f|, S) \leq M(f, S) - m(f, S). \quad (2)$$

**Proof.** Ex.

Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ be any partition of $[a,b]$.

By (2) \Rightarrow

$$M(|f|, [t_{k-1}, t_k]) - m(|f|, [t_{k-1}, t_k])$$

$$\leq M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])$$

$$\Rightarrow \sum_{k=1}^{n} \left( M(|f|, [t_{k-1}, t_k]) - m(|f|, [t_{k-1}, t_k]) \right) (t_k - t_{k-1})$$

$$= \sum_{k=1}^{n} \left( M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) \right) (t_k - t_{k-1})$$

$$\Rightarrow U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P).$$

For all $\varepsilon > 0$, there exists a partition $P$ such that

$$U(f, P) - L(f, P) < \varepsilon$$

$$\Rightarrow U(|f|, P) - L(|f|, P) < \varepsilon.$$

Thm 33.6

Let $f$ be a function defined on $[a,b]$. If $a < c < b$

and $f$ is integrable on $[a,c]$ and on $[c,b]$, then $f$ is
Let $f$ be a function defined on $[a,b]$. If $a < c < b$

and $f$ is integrable on $[a,c]$ and on $[c,b]$, then $f$ is

integrable on $[a,b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$ 

**Pf:** Read P285.

**Defn 33.7**

A function $f$ on $[a,b]$ is piecewise monotonic if there is

a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ of $[a,b]$ s.t

$f$ is monotonic on each interval $(t_{k-1}, t_k)$. The function $f$ is

piecewise continuous if $\exists$ a partition $P$ of $[a,b]$ s.t

$f$ is uniformly continuous on each $(t_{k-1}, t_k)$.

**Thm 33.8**

If $f$ is a piecewise continuous function or a bad piecewise

monotonic function on $[a,b]$, then $f$ is integrable on $[a,b]$.

**Pf:** Read P286.

**Thm 33.9**

If $f$ is continuous on $[a,b]$, then $\exists$ some $x \in (a,b)$ s.t

$$f(x) = \frac{1}{b-a} \int_a^b f.$$
Pf: Let $M$ and $m$ be the max and min of $f$ on $[a, b]$.

By Thm 18.1, $\exists x_0, y_0 \in [a, b]$ s.t $f(x_0) = m$ and $f(y_0) = M$

If $M = m$, then $f$ is a constant function and

$$f(x) = \frac{1}{b-a} \int_a^b f \text{ for all } x \in (a, b).$$

Now assume $M > m$. Note each function $M - f$ and $f - m$

is nonnegative and not identically 0. By Thm 33.4 (i),

$$\int_a^b m < \int_a^b f < \int_a^b M.$$

Thus

$$m < \frac{1}{b-a} \int_a^b f < M.$$

Then by I.V.T. $\exists x \in (a, b)$ between $x_0$ and $y_0$ s.t

$$f(x) = \frac{1}{b-a} \int_a^b f.$$