Problem 26.2

(a) Observe \( \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2} \) for \( |x| < 1 \).

\[ \text{Solution. Just see Example 1.} \]

(b) Evaluate \( \sum_{n=1}^{\infty} \frac{n}{3^n} \).

\[ \text{Solution. By (a), for } x = \frac{1}{2} \text{ we have } \sum_{n=1}^{\infty} n \left( \frac{1}{2} \right)^n = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2. \]

(c) Evaluate \( \sum_{n=1}^{\infty} \frac{n}{3^n} \) and \( \sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n} \).

\[ \text{Solution. By (a), for } x = \frac{1}{3} \text{ and } x = -\frac{1}{3} \text{ we have } \sum_{n=1}^{\infty} n \left( \frac{1}{3} \right)^n = \frac{\frac{1}{3}}{(1-\frac{1}{3})^2} = \frac{3}{4}, \text{ and } \sum_{n=1}^{\infty} n \left( -\frac{1}{3} \right)^n = \frac{-\frac{1}{3}}{(1+\frac{1}{3})^2} = -\frac{3}{16}. \]

Problem 26.4

(a) Observe \( e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \) for \( x \in \mathbb{R} \), since we have \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) for \( x \in \mathbb{R} \).

\[ \text{Solution. Just replace } x \text{ by } -x^2 \text{ in the expansion of } e^x. \]

(b) Express \( F(x) = \int_0^x e^{-t^2} dt \) as a power series.

\[ \text{Solution. Since the power series of } e^{-x^2} \text{ converges for all } x \in \mathbb{R}, \text{ we can integrate it term by term to get } F(x). \text{ Hence we have } F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1) \cdot n!} x^{2n+1}. \]

Problem 26.6 Let \( s(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \) and \( c(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots \) for \( x \in \mathbb{R} \).
(1) Prove $s' = c$ and $c' = -s$.

proof. Since the power series of $s$ and $c$ converges for $x \in \mathbb{R}$, we can differentiate both power serie term by term and get the result.

(2) Prove $(s^2 + c^2)' = 0$

proof. By (a), we have $(s^2 + c^2)' = 2ss' + 2cc' = 2sc - 2cs = 0$.

(3) Prove $s^2 + c^2 = 1$.

proof. Let $x = 0$, we have $s(x) = 0$, $c(x) = 1$. since (b) we know $s^2 + c^2$ is a constant, it must be 1.