Problem 29.16

**Proof.** Given \( y \in (-\pi/2, \pi/2) \), note that \( f'(y) = (\tan x)' = \frac{1}{\cos^2 y} \neq 0 \). So by Theorem 29.9, we have for \( x = f(y) \), \( g'(x) = g'(f(y)) = \frac{1}{f'(y)} = \cos^2(y) = \cos^2(\arctan x) \). Note that \( g'(y) \) is symmetric with respect to \( y \), so we only need to consider the case \( y \in (0, \pi/2) \).

Consider a right angle triangle with side lengths 1, \( x \) respectively and one angle \( y \) (like the figure below).

![Figure 1](image)

Note that \( \tan y = x \) and so \( \arctan x = y \).

So \( \cos y = \frac{1}{\sqrt{1+x^2}} \). So we have \( g'(x) = \cos^2(y) = \frac{1}{1+x^2} \).

\( \square \)

Problem 29.18

**Proof.** (a) Note that for any \( n > 0 \), we have \( |s_{n+1} - s_n| = |f(s_n) - f(s_{n-1})| \leq a|s_n - s_{n-1}| \leq a^n|s_1 - s_0| \), the middle inequality comes from the Mean Value Theorem. Given \( \epsilon > 0 \), because \( a^n \) goes to 0, there exists \( N \in \mathbb{N} \) s.t. for any \( n > N \), we have \( a^n < \frac{(1-a)\epsilon}{|s_1 - s_0|} \). Now consider any \( m \geq n \) greater than \( N \), we have

\[
|s_m - s_n| = |s_m - s_{m-1} + s_{m-1} \cdots s_{n+1} - s_n|
\leq |s_m - s_{m-1}| + |s_{m-1} - s_{m-2}| \cdots |s_{n+1} - s_n|
\leq a^{m-n-1}|s_{n+1} - s_n| + |s_{n+1} - s_n| \cdots |s_{n+1} - s_n|
\leq |s_{n+1} - s_n| \cdot (1 + a + a^2 \cdots + a^{m-n-1})
\leq |s_{n+1} - s_n| \cdot \frac{1}{1-a}
\leq \frac{a^n|s_1 - s_0|}{1-a}
< \epsilon
\]
That is to say, \( \{s_n\} \) is Cauchy, i.e., converges.

(b) By part (a), we have \( s = \lim_{n \to \infty} s_n = \lim_{n \to \infty} f(s_{n-1}) = \{\text{by the continuity of } f\} = f(\lim_{n \to \infty} s_{n-1}) = f(s). \)

**Problem 31.1**

**Proof.** Centering at \( x_0 = 0 \), compute \( f^{(n)}(x) \) we get it is either \( \pm \cos x \) or \( \pm \sin x \) (so it is all bounded by 1 on \( \mathbb{R} \)).

\[
a_n = \frac{f^{(n)}(0)}{n!} = \begin{cases} 
0 & n = 2k + 1 \\
\frac{(-1)^k}{(2k)!} & n = 2k
\end{cases}
\]

So by Theorem 31.4, we have \( \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \). \( \square \)

**Problem 31.2**

**Proof.** We could do the same induction to get \( f^{(n)}(0) \) like 31.1, or we can do the following steps. Note that \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \) and it converges uniformly on any bounded interval \([-M, M]\). So we have

\[
\sinh x = \frac{1}{2} (e^x - e^{-x}) = \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{(2n + 1)!} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n + 1)!}
\]

\[
\cosh x = \frac{1}{2} (e^x + e^{-x}) = \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} \right) = \sum_{n=0}^{\infty} \frac{2x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}
\]

\( \square \)

(Practice) **Problem 1**

**Proof.** Note that \( f^n(x) = e^x \) for all \( x \in \mathbb{R} \). So \( f^2(2) = e^2 \). So we get \( e^x = \sum_{n=0}^{\infty} \frac{e^2 \cdot (x - 2)^n}{n!} \).

\( \square \)

(Practice) **Problem 2**

**Proof.** Note that \( f(x) = e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} \) for all \( x \in \mathbb{R} \). And \( a_{10} = \frac{1}{5!} = f^{(10)}(0) \frac{0}{10!} \), so \( f^{(10)}(0) = \frac{10!}{5!} \).

\( \square \)
References