MATH 20D – Dr. Xiao
Section D07/D08
Ariel (Ari) Schreiman
6/1/21

Announcements

- TA Evaluations!
  - Separate from Course/Professor evaluation (CAPE)
  - Due Monday June 7th at 11:59pm PDT
- Homework 9 due this Sunday June 7 at 11:59pm PDT
- MATLAB Quiz tomorrow!
  - On Canvas under Quizzes
  - Review of MATLAB Homeworks 1-4
  - Open notes, book, and assignment pages
  - Not designed to be difficult, just to make sure you understood the concepts in each homework
- Finals week:
  - Will host a Final review session Tuesday at 6pm PDT
  - And one more flex office hour before the final

Example Problems

Find the general solution for the system: $\begin{matrix} x_1' = x_1 + 2x_2 \\ x_2' = 4x_1 + 3x_2 \end{matrix}$

$$\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\mathbf{x}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{rt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} 1-r & 2 \\ 4 & 3-r \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1-r)(3-r) - 8 = 0$$

$$r^2 - 4r - 5 = 0$$

$$(r-5)(r+1) = 0$$

$$r = 5, -1$$

$$\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2u_1 = u_2 = c_1$
$u_2 = c_1$

$$\begin{bmatrix} c_1/2 \\ c_1 \end{bmatrix} \rightarrow c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$u_1 + u_2 = 0$
$u_2 = c_2$
$u_1 = -c_2$

$$\begin{bmatrix} -c_2 \\ c_2 \end{bmatrix} = c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{5t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$x_1(t) = c_1 e^{5t} - c_2 e^{-t}$
$x_2(t) = 2c_1 e^{5t} + c_2 e^{-t}$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e^{5t} & -e^{-t} \\ 2e^{5t} & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Systems of Linear Differential Equations with Constant Coefficients

- Sometimes we might end up with a system of first-order differential equations that we want to solve (many reasons why this might be the case in physics, chemistry, biology, etc) and want to figure out how to solve it
- Such a system would look like this (for two variables, but there could be more):

$$x_1'(t) = ax_1(t) + bx_2(t) + g_1(t)$$
$$x_2'(t) = cx_1(t) + dx_2(t) + g_2(t)$$

- We can use matrix notation to write this system like this:

$$\mathbf{x}(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ \mathbf{g}(t) = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$
$$\frac{d}{dt}\mathbf{x}(t) = A\mathbf{x}(t) + \mathbf{g}(t)$$

- Alternatively we can take the derivative of the first equation and use substitution to get:

$$x_1''(t) = ax_1'(t) + b[cx_1(t) + dx_2(t) + g_2(t)] + g_1'(t) \tag{1}$$
$$x_1'' - ax_1' - bcx_1 = bdx_2 + bg_2 + g_1' \tag{2}$$

- Side note: we could go the opposite direction for a higher-order differential equation $a_n x^{(n)}(t) + a_{n-1}x^{(n-1)}(t) \ldots + a_1 x'(t) + a_0 x(t) + g(t) = 0$ by choosing $x_1 = x^{(n-1)} = x_2',\ x_2 = x^{(n-2)} = x_3',\ \ldots, x_{n-1} = x' = x_n', x_n = x$ and rearranging the original equation to get $a_n x_1' = -[a_{n-1}x_1 + a_{n-2}x_2 + \ldots + a_1 x_{n-1} + a_0 x_n + g(t)]$. We therefore get a system of n linear first-order differential equations with n unknown functions.

- What we can see is that we are actually solving a second-order nonhomogeneous differential equation for both x_1 and x_2
  - We should therefore expect to find solutions in similar ways
- The first method of solution we could use is Laplace transforms
  - Take the Laplace Transform of each equation, we can then solve for each individual function
- However, we can also use methods from linear algebra to solve our system!
  - Chapters 9.2 and 9.3 have a good overview of these techniques
- We will first attempt to solve the homogeneous equation $\mathbf{x}' = A\mathbf{x}$
  - We will assume solutions of the form $\mathbf{x} = e^{rt}\mathbf{u}$, just like we did for constant-coefficient homogeneous second-order differential equations
  - Taking the derivative, we get $\mathbf{x}' = re^{rt}\mathbf{u}$, and therefore $re^{rt}\mathbf{u} = Ae^{rt}\mathbf{u}$
  - Dividing by e^{rt}, we get $A\mathbf{u} = r\mathbf{u}$, which is called an eigenvalue equation for A
    * Values for r are called eigenvectors
    * Vectors $\mathbf{u}$ which solve the equation for a given eigenvector are called eigenvectors
    * From linear algebra, we find all eigenvalues by setting the determinant of the matrix $A - r\mathbf{I}$ ($\mathbf{I}$ represents the identity matrix) equal to zero, which is a polynomial called the *characteristic equation*, and their corresponding eigenvectors from $(A - r\mathbf{I})\mathbf{u} = \mathbf{0}$ ($\mathbf{0}$ is the zero vector)
    * For any eigenvalue, there will be infinite eigenvectors that solve the equation
      · Typically, we can represent all these eigenvectors by $c\mathbf{v}$, where c is an arbitrary constant and v is one particular eigenvector
      · However, if we have a repeated eigenvalue in the characteristic equation, we will have eigenvectors of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots$, where all the vectors are linearly independent
      · If the eigenvalue is complex, we will have a complex eigenvector of the form $\mathbf{v} = \mathbf{a} + \mathbf{b}i$, where a and b are vectors with real coefficients
    * Here is the general method we use to solve for the eigenvectors and eigenvalues in a system of two variables:

$$(A - r\mathbf{I})\mathbf{u} = 0$$

$$\begin{bmatrix} a-r & b \\ c & d-r \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(a-r)(d-r) - bc = 0$$

$r = r_1, r_2$

$(a - r_1)u_1 + bu_2 = 0$
$cu_1 + (d - r_1)u_2 = 0$
$u_2 = c_1$
$\rightarrow \mathbf{u}_{r_1} = c_1\begin{bmatrix} m \\ n \end{bmatrix}$

$(a - r_2)u_1 + bu_2 = 0$
$cu_1 + (d - r_2)u_2 = 0$
$u_2 = c_2$
$\rightarrow \mathbf{u}_{r_2} = c_2\begin{bmatrix} p \\ q \end{bmatrix}$

- Distinct real eigenvalues: solution is $\mathbf{x} = c_1\mathbf{u}_1 e^{r_1 t} + c_2\mathbf{u}_2 e^{r_2 t}$
- Repeated eigenvalue: solution is $\mathbf{x} = c_1\mathbf{u}e^{rt} + c_2 e^{rt}[t\mathbf{u} + \mathbf{v}]$ where $(A - r\mathbf{I})\mathbf{v} = \mathbf{u}$
  - We can derive the second solution through the assumption that it will be of the form $\mathbf{a}te^{rt} + \mathbf{b}e^{rt}$ and substituting into the original equation and we find that $\mathbf{a} = \mathbf{u}$, and $\mathbf{b} = \mathbf{v}$, where $\mathbf{v}$ satisfies $(A - r\mathbf{I})\mathbf{v} = \mathbf{u}$
- Imaginary eigenvalues: solution is $\mathbf{x} = c_1 e^{\alpha t}[\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t] + c_2 e^{\alpha t}[\mathbf{a}\sin\beta t + \mathbf{b}\cos\beta t]$, where the eigenvalues are $\alpha \pm \beta i$

- Now if we want to solve the nonhomogeneous system $\mathbf{x}' = A\mathbf{x} + \mathbf{g}$, we can either use undetermined coefficients of variation of parameters!
  - Solutions will be of the form $\mathbf{x} = \mathbf{X}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \mathbf{x}_p$, where $\mathbf{X}$ is a 2x2 matrix that contains the two linearly independent solutions for the corresponding homogeneous problem, and $\mathbf{x}_p$ is a vector (or a sum of vectors) that is a particular solution to the system
- Undetermined coefficients: we will follow very similar rules for test functions as we did previously with this method, then solve for the values of each vector
  - There are some slight differences, however, which I can explain in office hours if you are curious. They shouldn't come up in the problems for this class
- Variation of parameters: the textbook shows a nice derivation of this formula, which is pretty similar to the previous derivation for our second order equations, and the result is that $\mathbf{x}_p = \mathbf{X}\int \mathbf{X}^{-1}\mathbf{g}\, dt$, where $\mathbf{X}^{-1}$ is the inverse of $\mathbf{X}$ ($\mathbf{X}^{-1}\mathbf{X} = \mathbf{X}\mathbf{X}^{-1} = \mathbf{I}$)
  - For a 2x2 matrix, we can calculate $\mathbf{X}^{-1}$ directly: $\mathbf{X}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, for $\mathbf{X} = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}$