MATH 20D – Dr. Xiao
Section D05/D06
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Announcements

- Introduction
- Check out the Discord (link on the Canvas announcement or on the course website)
- Make sure you have access to the textbook and gradescope
  - There is a MATLAB gradescope and a separate course gradescope
- Homework 1 due this Friday (April 9) at 11:59pm PDT
- MATLAB Assignment 1 due next week (April 16)
  - Download MATLAB version 2021a soon!
- Discussion section format:
  - Come to either time (1pm or 2pm PDT) that is more convenient for you
  - Not mandatory to attend
  - I will not be recording these sessions
  - However, these notes will be edited and posted after the discussion sections
    - Expect these to be released Tuesday evenings
    - Will be accessible on Canvas under Files and on the course website under Lecture Notes
- Office Hours:
  - Thursday 1pm every week
  - Flexible hour
    - See *#office-hour-times* on the Discord for more info

Example Problems

1. Find all solutions to the equation $y' = y$
We can see that this is a separable differential equation.
First let's check if $y' = 0$ results in a solution:
Substituting, we get $y = 0$. This function does indeed have a derivative of 0 and therefore solves this equation.
Now, let's restrict ourselves to functions where $y \neq 0$. In this case we can divide both sides of the equation by y and get:
$\frac{1}{y} * \frac{dy}{dx} = 1$. Since y is a function of x, the expression $\frac{d}{dx}(F(y)) = f(y) * \frac{dy}{dx}$ by the chain rule, with $F(y) = \int f(y)dy$.
Therefore, if we take the integral of both sides of $\frac{1}{y} * \frac{dy}{dx} = 1$ in terms of x, we get:

$$\int \frac{1}{y}\, dy = \int 1 * dx \rightarrow \ln|y| = x + c \rightarrow e^{\ln|y|} = e^{x+c} \rightarrow |y| = e^c * e^x \rightarrow y = \left(\pm e^c\right) e^x$$

Notice how we carry the constant c throughout our simplification of our implicit solution when finding the explicit solution for this problem. Also, for any given initial value of y, we will have exactly one constant value for $\pm e^c$, so we can define $C = \pm e^c$ and rewrite our solution as $y = Ce^x$ for simplicity. Since for all $C \neq 0$ this equation results in a function where $y \neq 0$ for all x, we can confirm that this equation fits our initial restriction.

2. Find all solutions for $x\frac{dy}{dx} + 2y = 3$

We can see that this is a linear first-order ODE, so first we want to put this equation into the standard form: $\frac{dy}{dx} + \left(\frac{2}{x}\right)y = \frac{3}{x}$. We therefore must assume that $x \neq 0$.

Now let's find our integrating factor $\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^{\ln|x|^2} = |x|^2 = x^2$.

Therefore, $y = \dfrac{\int x^2\left(\frac{3}{x}\right)dx + c}{x^2} = \frac{3}{2} + \frac{c}{x^2}$.

As we can see, this solution is valid for all values of $x \neq 0$.

Typically for a problem like this we will pick some inital value where $x > 0$, and can restrict the domain of y to positive values of x, since y is continuous and differentiable on the entire domain.

3. Find all solutions to $\left(ye^{xy} + 2\right) + \left(xe^{xy}\right)\frac{dy}{dx} = 0$

This is a nonlinear equation and it is not separable either, but it might be an exact equation. We have to check if we can find a function $F(x, y)$ such that $ye^{xy} + 2$ is the partial derivative in terms of x and xe^{xy} is the partial derivative in terms of y.

We can take integrals of both functions in terms of x and y, respectively, and see if there is a function $F(x, y)$ that satisfies both integrals.

$$\int \left(ye^{xy} + 2\right)dx = e^{xy} + 2x + a(y)$$

$$\int xe^{xy}dy = e^{xy} + b(x)$$

Notice that instead of just adding a constant of integration c, we must add an arbitrary function a(y) or function b(x) because $\frac{\partial}{\partial x}a(y) = 0$ and $\frac{\partial}{\partial y}b(y) = 0$.

Here, we can choose $a(y) = c$ and $b(x) = 2x + c$ and therefore get the implicit solution:

$F(x, y) = e^{xy} + 2x + c = 0$

If we wanted to find a solution for an initial value problem, we would just substitute the x and y values into the above equation and solve for c.

Calculus Review

- Implicit Differentiation
  - We have an expression in terms of x and y where y is a function of x
  - Therefore we must use the chain rule and/or product rule when finding the derivative in terms of x
  - For example, let's implicitly differentiate $e^{xy} + y = 2$:
    * $\frac{d}{dx}\left(e^{xy} + y\right) = \frac{d}{dx}(2)$
    * $[y + y'x]*e^{xy} + y' = 0$
- Partial Derivatives
  - We have an expression in terms of multiple variables, but we are taking the derivative of only one variable at a time. We treat all other variables like constants.
  - For example, let's find the partial derivatives of $e^{xy} + y$
    * $\frac{\partial}{\partial x}\left(e^{xy} + y\right) = ye^{xy}$, $\frac{\partial}{\partial y}\left(e^{xy} + y\right) = xe^{xy} + 1$
    * Notice that these partial derivatives are not the same as the implicit differentiation above!
- Integration Techniques
  - u-Substitution
    * When integrating an expression of the form $\int f'(x)*g(f(x))dx$, we can define $u = f(x)$. Therefore, $du = f'(x)dx$, and we can substitute to get $\int f'(x)*g(f(x))dx = \int g(u)du$ to solve.
    * For a definite integral, you must remember to convert the bounds of integration from $[x_0, x_1]$ to $[u_0 = f(x_0), u_1 = f(x_1)]$
    * Example: $\int_{x=1}^{x=2} e^{x+5}\, dx \to u(x) = x + 5 \to du = dx \to \int_{u=6}^{u=7} e^{u}\, du = e^7 - e^6$
  - Integration by Parts
    * When integrating an expression of the form $\int a(x)*b(x)dx$, we can take advantage of the product rule for differentiation to find a simpler integral to evaluate:
      · $\frac{d}{dx}[u(x)*v(x)] = u'v + uv'$
      · Rearranging and integrating gives us: $\int uv'\, dx = uv - \int u'v\, dx$
      · So if we can make suitable choices for $u(x)$ and $v'(x)$ such that the derivative of u is a function that is simpler to integrate and/or the function v is simpler to integrate (or at least not more complicated), then the integral on the right side of this equation should be easier to solve.
      · Sometimes you will need to check multiple possibilities for u and v
    * Example:

- $\int xe^x dx \rightarrow u = x, v' = e^x \rightarrow u' = 1, v = e^x$
- So $\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c$

Differential Equations Terminology

- What is a differential equation?
  - Equation that relates derivatives of a function or functions
    - Usually found when discussing physical processes: for example, air resistance is a change in acceleration that is a function of the velocity of an object
  - Ordinary differential equations only involve functions of one variable – usually $y(x)$
  - Partial differential equations can relate functions of multiple variables
    - We will not be covering these equations in this class
  - Typically, we want to find a solution to a differential equation
    - A function that, when substituted into the original equation, makes that equation true – it can be simplified to $0 = 0$
  - For ordinary differential equations, we can represent them with either:
    - Leibniz notation: $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$
    - Prime notation: $y', y'', \ldots$
  - The *order* of an ODE is the value of the highest derivative in the equation
    - A first order equation only has $\frac{dy}{dx}$, while a second order equation includes a $\frac{d^2y}{dx^2}$ term, etc
- Linear/Non-linear ODE
  - A differential equation of the form $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} \ldots a_1y' + a_0(x)y = g(x)$ is called *linear* and we have special techniques to help solve these equations
  - Anything else is nonlinear, these are typically more difficult to solve
- Implicit/Explicit Solution
  - A solution to an ODE of the form $f(x) = y$ is called an explicit solution
  - A solution to an ODE of the form $F(x, y) = 0$ is called an implicit solution
    - Sometimes we can't solve for y
- Initial Value Problems
  - Differential equations can have infinitely many solutions
  - Once we define initial conditions, that narrows down the number of possible solutions
  - For the functions in this class, typically this can be reduced to only one unique solution
    - See theorem 1 in the textbook for a special case

Separable Differential Equations

- Formula
  - $y' = \frac{a(x)}{b(y)}$

- Method of Solution
  - First check if $y' = 0$, solve for y and see if this equation is a solution
    - For example $y' = y^2 - 4$ has the solutions $y = 2,\ y = -2$
  - Then we can rearrange the equation to: $b(y)dy = a(x)dx$ and integrate

Linear First-Order Equations

- Formula
  - $a_1(x)y' + a_2(x)y = g(x)$
  - First rearrange by dividing by $a_1(x)$ on both sides, making sure it is not zero
    - This becomes the standard form $y' + P(x)y = Q(x)$
- Method of Solution
  - We want to find a function $\mu(x)$ such that $[\mu * y]' = \mu * y' + y * \mu' = \mu * y' + y * \mu * P(x) = \mu[y' + P(x)y]$
  - Therefore, if we multiply both sides of the equation by $\mu(x)$, we can easily integrate both sides in terms of x and then divide by $\mu(x)$ to find an explicit solution
    - This means $\mu(x) \neq 0$ must be true
  - The choice $\mu(x) = e^{\int P(x)dx}$ satisfies both of these conditions
  - So $y = \dfrac{\int \mu(x)Q(x)dx\ +\ C}{\mu(x)}$

Exact Equations

- Formula
  - $M(x, y) + N(x, y)\dfrac{dy}{dx} = 0$

- Method of Solution
  - If we have a nonlinear ODE with $M(x, y) = \dfrac{\partial}{\partial x}F(x, y)$ and $N(x, y) = \dfrac{\partial}{\partial y}F(x, y)$, then the derivative in terms of x of some equation $F(x, y) = c$ provides us with the formula above.
  - So we can check $\int M(x, y)dx = \int N(x, y)dy = F(x, y)$
  - Then solve for y, if possible, to get an explicit solution