MATH 20D – Dr. Xiao
Section D05/D06
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Announcements

- Midterm on Friday, see the exam instructions: http://www.math.ucsd.edu/~m3xiao/math20d/Instruction-Exam.pdf
  - 11am or 6pm PDT, 65 minutes
  - Open-notes but not open-internet
- Review session tomorrow at 6pm PDT
  - Zoom link: https://ucsd.zoom.us/j/96758094578 (also on canvas announcement)
  - Make sure to look at the practice problems and Prof. Xiao's review lecture
- MATLAB Homework due Friday at 1pm PDT
- Homework 2 due Sunday at 11:59pm PDT

Example Problems

1. Solve $\left(x^2+y^2-5\right)+(y+xy)y'=0$ with initial condition $y(0)=1$

We start by checking to see if this is already an exact equation:

$$\int \left(x^2+y^2-5\right)dx = \frac{x^3}{3}+xy^2-5x+g(y)$$

$$\int (y+xy)dy = \frac{y^2}{2}+\frac{xy^2}{2}+h(x)$$

Since it is not, we need to use an integrating factor.
$M_y = 2y,\ N_x = y$

We can start by checking if $\mu = f(y) \rightarrow \dfrac{N_x - M_y}{M} = \dfrac{y-2y}{x^2+y^2-5} = \dfrac{-y}{x^2+y^2-5}$, which doesn't work.
But if we test $\mu = f(x) \rightarrow \dfrac{M_y - N_x}{N} = \dfrac{2y-y}{y+xy} = \dfrac{y}{y(x+1)} = \dfrac{1}{x+1}$ then this results in the separable differential equation $\dfrac{d\mu}{dx} = \dfrac{1}{x+1}\mu$ so $\mu(x) = e^{\int \frac{1}{x+1}dx} = e^{\ln|x+1|} = |x+1|$. Since our initial condition implies that $x \in [0,\infty)$, then $x+1>0$, so we can simplify to $\mu(x) = x+1$.

Now all we have to do is multiply by $\mu(x)$ and solve our exact equation:

$$\left(x^2+y^2-5\right)(x+1)+(y+xy)(x+1)y' = 0 \quad (1)$$

$$\left[x^3+x^2-5x-5+y^2(x+1)\right]+\left[y\,(x+1)^2\right]y' = 0 \quad (2)$$

$$\int \left(x^3+x^2-5x-5+y^2(x+1)\right)dx = \frac{x^4}{4}+\frac{x^3}{3}-\frac{5x^2}{2}-5x+\frac{(x+1)^2y^2}{2}+g(y) \quad (3)$$

$$\int y\,(x+1)^2 dy = \frac{(x+1)^2 y^2}{2} + h(x) \tag{4}$$

$$g(y) = c;\; h(x) = \frac{x^4}{4} + \frac{x^3}{3} - \frac{5x^2}{2} - 5x + c \tag{5}$$

$$F(x,y) = \frac{x^4}{4} + \frac{x^3}{3} - \frac{5x^2}{2} - 5x + \frac{(x+1)^2 y^2}{2} + c = 0 \tag{6}$$

$$y^2 = \frac{-3x^4 - 4x^3 + 30x^2 + 60x - 12c}{6(x+1)^2} \tag{7}$$

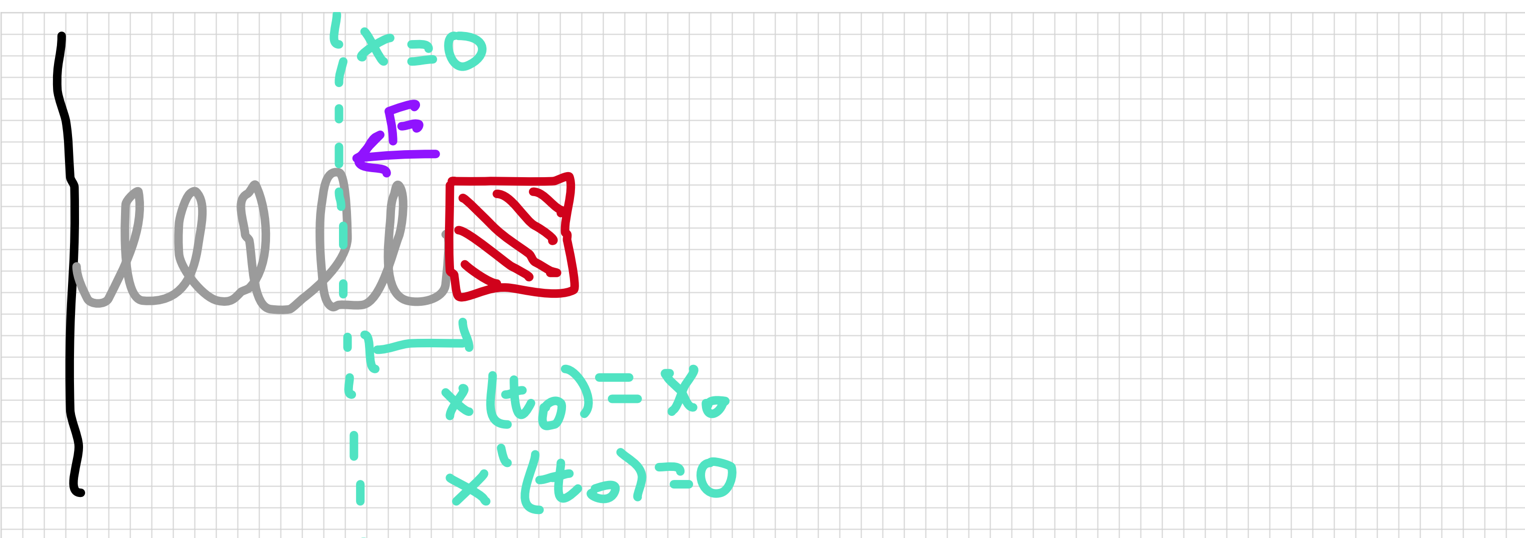
$$y = \pm\sqrt{\frac{-3x^4 - 4x^3 + 30x^2 + 60x - 12c}{6(x+1)^2}} \tag{8}$$

And we can plug in our initial condition to solve for c:

$$1 = \pm\sqrt{\frac{-12c}{6}} = \pm\sqrt{-2c} \rightarrow c = -\frac{1}{2}$$

So our final solution is: $y = \sqrt{\dfrac{-3x^4 - 4x^3 + 30x^2 + 60x + 6}{6\,(x+1)^2}}$

2. A. Find the equation of motion for an ideal spring with spring constant k and a mass m attached to the end, with initial position x_0 and zero initial velocity.



For an ideal spring, $F = -kx$. Remembering Newton's second law, $F = ma = mx''$, we get the second-order homogeneous equation: $mx'' + kx = 0$. If we assume this equation will have solutions of the form $x = e^{\lambda t}$, then the characteristic equation is $m\lambda^2 + k = 0$ and $\lambda = \pm\, i\sqrt{\frac{k}{m}}$.

Therefore, the general solution is $x(t) = c_1 \cos\left(t\sqrt{\frac{k}{m}}\right) + c_2 \sin\left(t\sqrt{\frac{k}{m}}\right)$. Since $x(0) = x_0$ and $x'(0) = 0$, we get $c_1 = x_0,\ c_2 = 0$. Our equation of motion is $x(t) = x_0 \cos\left(t\sqrt{\frac{k}{m}}\right)$.

B. Now let's add some dampening. No spring is perfectly ideal so there is some resistance to motion, or friction, which is a force that is proportional (and opposing) the velocity of the object. So we must include $F_{friction} = -bv = -bx'$, where b is a constant that represents the amount of internal friction in the spring.

Our equation of motion becomes $mx'' + bx' + kx = 0$, and so the characteristic equation

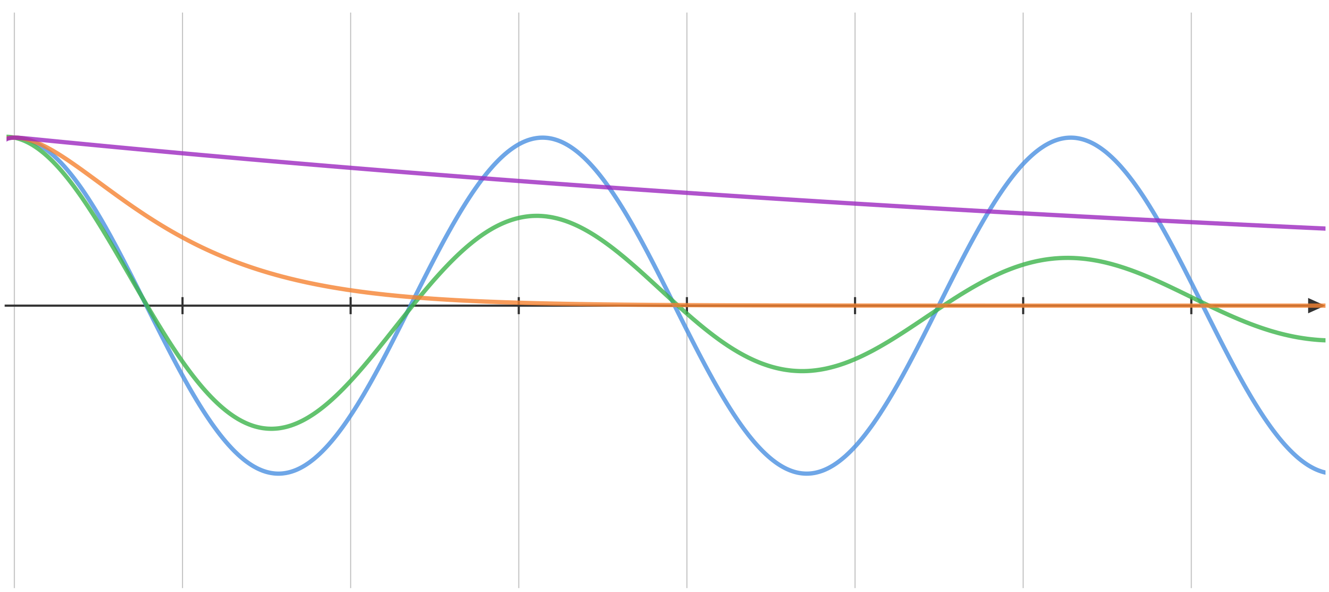
$m\lambda^2 + b\lambda + k = 0$. The solutions for lambda are: $\lambda = -\frac{b}{2m} \pm \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$.

This means there are three possible cases for our damped spring:

- "Underdamped": $\frac{k}{m} > \left(\frac{b}{2m}\right)^2$. This is the typical situation where the spring has a small value for b and therefore we still have two imaginary roots. Our general solution is $x(t) = e^{-\frac{b}{2m}t}\left[c_1 \cos\left(t\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}\right) + c_2 \sin\left(t\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}\right)\right]$. For the initial conditions from part A, our particular solution is $x(t) = x_0\left(e^{-\frac{b}{2m}t}\right)\cos\left(t\sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}\right)$.
- "Critically damped": $\frac{k}{m} = \left(\frac{b}{2m}\right)^2$. This is a situation where the spring is damped very specifically so that it is just on the border between vibrational motion and nonvibrational motion. Our general solution is $x(t) = (c_1 + c_2 t)e^{-\frac{b}{2m}t}$ and for our initial conditions the equation of motion is $x(t) = x_0\left(1 + \frac{b}{2m}t\right)e^{-\frac{b}{2m}t}$.
- "Overdamped": $\frac{k}{m} < \left(\frac{b}{2m}\right)^2$. This is typically the case when the spring constant is small or the mass of the object is large. The damping is so strong that the spring doesn't even vibrate, it just gradually moves towards the equilibrium position! The general solution is

$x(t) = e^{\left(-\frac{b}{2m}+\sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}\right)t}\left[c_1 + c_2 e^{\left(-2\sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}\right)t}\right]$ and our equation of motion given the intial conditions is $x(t) = x_0 e^{\left(-\frac{b}{2m}+\sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}\right)t}\left[\frac{b+\sqrt{b^2-4km}}{2\sqrt{b^2-4km}} - \left(\frac{b-\sqrt{b^2-4km}}{2\sqrt{b^2-4km}}\right)e^{\left(-2\sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}\right)t}\right]$.

If we choose $x_0 = 2,\ k = 1,\ m = 1, b_{under} = 0.2, b_{critical} = 2, b_{over} = 20$, then we can compare the undamped spring and the three damped cases:



Purple: overdamped, Orange: critically damped, Green: underdamped, Blue: undamped

Special Integrating Factors

- Formula:
  - $M(x,y) + N(x,y)\, y' = 0$
- Method of solution:
  - We want to multiply by a function $\mu(x,y)$ such that this equation becomes exact ($\mu \neq 0$ is a necessary restriction)

- For this to work, $\frac{\partial}{\partial y}[\mu(x, y)\, M(x, y)] = \frac{\partial}{\partial x}[\mu(x, y)\, N(x, y)]$ must be true
    - This is because of the equality of mixed partial derivatives – we are finding a function F such that μM is the x partial and μN is the y partial. So the y derivative of μM must be equal to the x derivative of μN, as both are equal to $\frac{\partial^2 F}{\partial x\, \partial y}$
- We can use the product rule to differentiate the above equation, however this gives us a very tricky partial differential equation to solve
- But there are two special cases:
    - μ is only a function of x and $\frac{M_y - N_x}{N}$ is also solely a function of x
    - μ is only a function of y and $\frac{N_x - M_y}{M}$ is also solely a function of y
- In these cases, we get a separable differential equation and can integrate, yielding:
    - $\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$
    - $\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$
- Then we can multiply our original equation by μ and solve the new exact equation as we have already learned

Homogeneous Linear Second-Order ODEs with Constant Coefficients

- Existence of solutions:
    - For a linear initial value problem $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x);\ y(x_0) = y_0,\ y'(x_0) = y_1$, a solution exists and is unique
    - Fundamental set of solutions
        - For a second-order equation, we will have exactly two *linearly independent* solutions.
            - This means that the two solutions are not just multiples of each other
            - $c_1y_1 + c_2y_2 \neq 0$
        - Because $\frac{d}{dx}[y_1 + y_2] = {y_1}' + {y_2}'$ (the derivative function is linear), the sum of these two linearly independent solutions (multiplied by any arbitrary constants) is also a solution to the equation
        - For any set of initial conditions, this composite function will be a solution to the differential equation
        - By our theorem above, we conclude that this solution is unique
    - General solution
        - The composite of two linearly independent solutions y_1 and y_2 for a second-order linear ODE: $y = c_1y_1 + c_2y_2$ is called the *general solution* to the equation
- Homogeneous equations:
    - Linear equations where $g(x) = 0$
    - Eventually, we will see how we can solve nonhomogeneous equations in a very similar way to the homogeneous equations

- We will start with the simplest case – homogeneous, linear, second-order ODEs with constant coefficients
- Equation:
  - $ay'' + by' + cy = 0,\ a \neq 0$
- Method of solution:
  - We will start by assuming that this equation will have a solution of the form $y = e^{mx}$
  - $y' = me^{mx};\ y'' = m^2e^{mx}$
  - Therefore, our equation becomes $am^2e^{mx} + bme^{mx} + ce^{mx} = 0$
  - Since $e^{mx} \neq 0$ for all m and x, we can divide both sides by e^{mx}
  - This gets us the *characteristic equation*: $am^2 + bm + c = 0$
  - Use quadratic formula $m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve for m
    - Don't ignore imaginary roots!
  - Three cases for our solution:
    - Two real roots: general solution is $y = c_1e^{m_1x} + c_2e^{m_2x}$
    - One real root: general solution is $y = c_1e^{mx} + c_2xe^{mx}$
      - For now, we won't prove how we "guessed" that xe^{mx} is our second solution, but we will revisit this later on
    - Imaginary roots:
      - Since a, b, and c are real coefficients, our imaginary solutions will be of the form $m = \alpha \pm \beta i$ (conjugate pair)
      - Therefore, our general solution is $y = c_1e^{(\alpha+\beta i)x} + c_2e^{(\alpha-\beta i)x} = e^{\alpha x}\left[c_1e^{i\beta x} + c_2e^{-i\beta x}\right]$
      - By using Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$, and choosing appropriate constants for c_1 and c_2, we can determine that $y = e^{\alpha x}\cos\beta x$ and $y = e^{\alpha x}\sin\beta x$ are both real solutions to the differential equation
      - We can rewrite our general solution as $y = e^{\alpha x}[k_1\cos\beta x + k_2\sin\beta x]$