MATH 20D – Dr. Xiao
Section D05/D06
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Announcements

- Good job on the midterm!
  - Should be graded by Friday
  - Common mistakes so far:
    - Remembering how to solve certain integrals
    - Where to put the +c
  - See announcement about the grading for problem 4(b)
- Homework 3 due this Sunday
  - Only includes Method of Undetermined Coefficients

Example Problems

1. Find the general solution to $y'' + 2y' + y = x^2e^{-x}$

We notice that the left side of this equation has constant coefficients, and the right hand side is a function which can be annihilated. Therefore we can use the method of undetermined coefficients to solve this problem.

First let's solve for y_h, the general solution to the homogeneous equation: $y_h'' + 2y_h' + y_h = 0$. This has the characteristic equation: $m^2 + 2m + 1 = (m+1)^2 = 0 \to m = -1$, so $y_h = c_1e^{-x} + c_2xe^{-x}$

Now, let's determine the necessary differential operator to annihilate $g(x) = x^2e^{-x}$. We see that
$(D+1)^3g = g''' + 3g'' + 3g' + g =$
$\left(-6e^{-x} + 6xe^{-x} - x^2e^{-x}\right) + 3\left(2e^{-x} - 4xe^{-x} + x^2e^{-x}\right) + 3\left(2xe^{-x} - x^2e^{-x}\right) + \left(x^2e^{-x}\right) = 0$
Which means that applying the differential operator $(D+1)^3$ to both sides of our nonhomogeneous equation will yield a linear homogeneous fifth-order with constant coefficients:
$(D+1)^2y = x^2e^{-x} \to (D+1)^3(D+1)^2y = (D+1)^3\left(x^2e^{-x}\right) \to (D+1)^5y = 0$
This fifth-order equation has the general solution:
$y = \left[c_1e^{-x} + c_2xe^{-x}\right] + \left(c_3x^2e^{-x} + c_4x^3e^{-x} + c_5x^4e^{-x}\right) = y_h + y_p$ which is also the general solution to our original differential equation, for some undetermined coefficients c_3, c_4, c_5.

For convenience, let's use $c_3 = A, c_4 = B, c_5 = C$ and solve by substituting y_p into our original differential equation:
$y_p = Ax^2e^{-x} + Bx^3e^{-x} + Cx^4e^{-x} = e^{-x}\left[Ax^2 + Bx^3 + Cx^4\right]$
$y_p' = -e^{-x}\left[Ax^2 + Bx^3 + Cx^4\right] + e^{-x}\left[2Ax + 3Bx^2 + 4Cx^3\right]$
$= e^{-x}\left[2Ax + (3B - A)x^2 + (4C - B)x^3 - Cx^4\right]$

$y_p{}'' =$
$-e^{-x}\left[2Ax + (3B - A)x^2 + (4C - B)x^3 - Cx^4\right] + e^{-x}\left[2A + 2(3B - A)x + 3(4C - B)x^2 - 4Cx^3\right]$
$= e^{-x}\left[\,2A + (6B - 4A)x + (12C - 6B + A)x^2 + (-8C + B)x^3 + Cx^4\right]$

$\to\; y_p{}'' + 2y_p{}' + y_p = x^2e^{-x} \to \;\to \left[2A + 6Bx + 12Cx^2\right]e^{-x} = x^2e^{-x}$

$\to 2A + 6Bx + 12Cx^2 = 1x^2 + 0x + 0 \to 2A = 0,\; 6B = 0,\; 12C = 1 \to A = 0,\; B = 0,\; C = \dfrac{1}{12}$

$\therefore y_p = \dfrac{x^4e^{-x}}{12}$

So our general solution is $y = \left[c_1e^{-x} + c_2xe^{-x}\right] + \dfrac{x^4e^{-x}}{12}$

2. Find the general solution to $y'' - 2y' + 5y = e^x \sin x$
Firstly we can solve the homogeneous equation $y_h{}'' - 2y_h{}' + 5y_h = 0 \to y_h = e^x[c_1 \cos 2x + c_2 \sin 2x]$

Next we want to find a differential operator which annihilates $e^x \sin x$. We find that $\left(D^2 - 2D + 2\right)$ is the suitable choice.

$\left(D^2 - 2D + 5\right)y = e^x \sin x \to \left(D^2 - 2D + 2\right)\left(D^2 - 2D + 5\right)y = 0$
$\to y = e^x[c_1 \cos 2x + c_2 \sin 2x] + e^x[c_3 \cos x + c_4 \sin x] = y_h + y_p$
$\therefore y_p = e^x[A \cos x + B \sin x]$

Now we solve for the undetermined coefficients A and B:
$y_p = e^x[A \cos x + B \sin x]$
$y_p{}' = e^x[A \cos x + B \sin x] + e^x[-A \sin x + B \cos x] = e^x[(A + B)\cos x + (B - A)\sin x]$
$y_p{}'' = e^x[2B \cos x - 2A \sin x]$
$\to y_p{}'' - 2y_p{}' + 5y_p = e^x \sin x \to \;\to e^x[3A \cos x + 3B \sin x] = e^x \sin x$
$\to 3A \cos x + 3B \sin x = \sin x \to 3A = 0,\; 3B = 1 \to\; A = 0,\; B = \dfrac{1}{3}$
$\therefore y_p = \dfrac{e^x \sin x}{3}$
So the general solution is $y = e^x\left[c_1 \cos 2x + c_2 \sin 2x + \dfrac{\sin x}{3}\right]$

3. Find the general solution to $y'' - 2y' - 3y = 4e^x - 9$
$\left(D^2 - 2D - 3\right)y = 4e^x - 9 \to (D - 3)(D + 1)y = 4e^x - 9$
$\therefore y_h = c_1e^{3x} + c_2e^{-x}$
To annihilate $4e^x - 9$ we need to use two differential operators. $(D - 1)$ annihilates $4e^x$ and D annihilates

-9 (any constant). So we will use $D(D-1)$ to annihilate both:
$D(D-1)(D-3)(D+1)y = 0 \to y = c_1e^{3x} + c_2e^{-x} + c_3e^x + c_4 = y_h + y_p$
$\to y_p = Ae^x + B,\ y_p{}' = Ae^x,\ y_p{}'' = Ae^x \to y_p{}'' - 2y_p{}' - 3y_p = 4e^x - 9$
$\to Ae^x - 2Ae^x - 3Ae^x - 3B = 4e^x - 9 \to -4Ae^x - 3B = 4e^x - 9 \to A = -1,\ B = 3$
$\therefore y_p = -e^x + 3$

So the general solution is $y = c_1e^{3x} + c_2e^{-x} - e^x + 3$

4. Find the general solution to $y'' + y = \sec x$
This is a sneak peak at what we will discuss next week, you will cover this material during Friday's lecture so this and the explanation for Variation of Parameters below are here to help you understand that material if you want to get a head-start. You do not need to know this material to solve the problems on the homework this week.

When solving this problem, we notice that $g(x) = \tan x$ is not a function we can annihilate, meaning that we cannot use the method of undetermined coefficients. However, the corresponding homogeneous equation $y_h{}'' + y_h = 0$ is easy to solve so we can attempt to use variation of parameters to solve our nonhomogeneous equation.

First solve for y_h: $y_h{}'' + y_h = 0 \to y_h = c_1 \sin x + c_2 \cos x$

Therefore, $y_p = u_1(x)\sin x + u_2(x)\cos x$.

Using the equations for u_1, u_2 that we found in our section on variation of parameters we get:

$$u_1 = -\int \frac{y_2 f(x)}{y_1y_2{}' - y_1{}'y_2}dx = \int \frac{\cos x \sec x}{\sin^2 x + \cos^2 x}dx = \int \cos x \sec x\, dx = \int dx = x$$

$$u_2 = \int \frac{y_1 f(x)}{y_1y_2{}' - y_1{}'y_2}dx = \int \frac{\sin x \sec x}{-\sin^2 x - \cos^2 x}dx = -\int \sin x \sec x\, dx$$

$$= -\int \tan x\, dx = \ln|\cos x|$$

So $y_p = x \sin x + (\ln|\cos x|)\cos x$ and the general solution is:
$y = c_1 \sin x + c_2 \cos x + x \sin x + \cos x \ln|\cos x|$

Undetermined Coefficients

- Superposition principle
  - If $y_h = c_1y_1 + c_2y_2$ is the solution to the homogeneous linear second order equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, then the general solution to the nonhomogeneous linear equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$ will be of the form $y = y_h + y_p$, where y_p is a particular solution to the nonhomogeneous differential equation
  - So to find a solution to the nonhomogeneous problem we need to solve for both y_h and y_p

- Annihilator Operators
  - For a linear differential equation with constant coefficients $a_n y^{(n)} + a_{n-1}y^{(n-1)} \ldots a_1 y' + a_0 y = g(x)$, we can represent this as $L(y) = g(x)$, where $L = a_n D^n + a_{n-1}D^{n-1} \ldots a_1 D + a_0$ is a differential operator representing all the derivatives of y present in our equation
    - Because the derivative of the sum of two functions is equal to the sum of the derivatives of each, we can show that:
    - $L(y_h + y_p) = L(y_h) + L(y_p) = 0 + L(y_p) = g(x)$
      - Remember that y_h is the solution to the homogeneous equation $L(y_h) = 0$
    - This is a simple explanation for why our general solution to $L(y) = g(x)$ is of the form $y = y_h + y_p$
  - We can factor L into multiple operators of lower order: $L = (D - r_1)(D - r_2) \ldots (D - r_n)$, where each r represents a root of the polynomial.
    - We are allowed to do this because we are only changing the order which we are applying the derivatives, which we can do because derivatives are commutative
  - Example: $y'' + 3y' + 2y = 0$ can be represented as $\left(D^2 + 3D + 2\right)y = 0$, and this can be factored to yield $(D+1)(D+2)y = 0$. As we learned before, the general solution to this equation is $y = c_1 e^{-x} + c_2 e^{-2x}$.

    - Let's look at $y = e^{-x}$. $(D+1)y = \dfrac{dy}{dx} + y = -e^{-x} + e^{-x} = 0$
    - We say that $D + 1$ annihilates (makes equal to zero) e^{-x}
    - Similarly, $D + 2$ annihilates e^{-2x}
  - We can find that:
    - D^n annihilates the functions $1,\ x,\ x^2,\ \ldots x^{n-1}$
    - $(D - r)^n$ annihilates the functions $e^{rx},\ xe^{rx},\ \ldots x^{n-1}e^{rx}$
    - $\left[D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)\right]^n$ annihilates the functions $e^{\alpha x}\cos\beta x,\ xe^{\alpha x}\cos\beta x,\ \ldots x^{n-1}e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x,\ xe^{\alpha x}\sin\beta x,\ \ldots x^{n-1}e^{\alpha x}\sin\beta x$
      - Notice that the roots of $\left[D^2 - 2\alpha D + \left(\alpha^2 + \beta^2\right)\right]$ are $\alpha \pm \beta i$
  - So the general solution to any differential equation of the form $L(y) = 0$ will be the sum of all the possible annihilated functions, multiplied by constants
    - Example: $y''' + y'' = 0 \rightarrow D^2(D+1)y = 0$ has the general solution: $y = c_1 + c_2 x + c_3 e^{-x}$
  - Therefore, if we can find a differential operator L_2 that annihilates $g(x)$ in the equation $L_1(y) = g(x)$, then we can solve for the general solution of the equation by applying this operator to both sides, getting $L_2 L_1(y) = 0$, which we know how to solve
    - This general solution to $L_2 L_1(y) = 0$ will be of the form $y = [c_1 y_1 + c_2 y_2] + (c_3 y_3 + \ldots)$, where $y_1,\ y_2$ are the two homogeneous solutions. So to find the coefficients of $y_p = (c_3 y_3 + \ldots)$ we will solve $a_2 {y_p}'' + a_1 {y_p}' + a_0 y_p = g(x)$.

Variation of Parameters

- Reduction of order
  - Let's say we have one solution y_1 for a second-order linear homogeneous equation in standard form $y'' + P(x)y' + Q(x) = 0$
  - Now let's assume that $y_2 = u(x)*y_1$ for some function $u(x)$
  - Taking derivatives and simplifying, we get $y_1u'' \; + \; (2y_1{'} + Py_1)u' = 0$
  - If we choose the substitution $w = u'$, then we get a separable first-order equation that we can solve
  - Ultimately, we get $u(x) = \displaystyle\int \frac{e^{-\int P(x)dx}}{[y_1(x)]^2}dx$
  - Example: We know that $y_1 = e^{-x}$ is a solution to the equation $y'' + 2y' + y = 0$, which has repeated roots. What is the second linearly-independent solution to this equation?
    * We get $u(x) = \displaystyle\int \frac{e^{-\int 2dx}}{\left[e^{-x}\right]^2}dx = \int \frac{e^{-2x}}{e^{-2x}}dx = \int dx = x$, so $y_2 = xe^{-x}$, which is what we had previously found to be true!
  - The bottom line here is that multiplying one solution by a function of x can help to solve for a second solution. *You won't be asked to use this method on a test, I am just including it here because it helps explain our rationale below (which you will need to apply).*
- So now we can develop a general method for finding the particular solution y_p for a nonhomogeneous equation $y'' + P(x)y' + Q(x)y = g(x)$ where $y = c_1y_1 + c_2y_2$ is the general solution of the corresponding homogeneous equation $y'' + P(x)y' + Q(x)y = 0$.
  - We will make a similar assumption as we did above, and assume that $y_p = u_1(x)*y_1 + u_2(x)*y_2$ for some functions $u_1,\; u_2$
  - Taking the derivatives and substituting: $y_p{''} + P(x)y_p{'} + Q(x)y_p = g(x)$, then simplifying, we get: $\dfrac{d}{dx}[y_1u_1{'} + y_2u_2{'}] + P[y_1u_1{'} + y_2u_2{'}] + (y_1{'}u_1{'} + y_2{'}u_2{'}) = f(x)$
  - If we choose $y_1u_1{'} + y_2u_2{'} = 0$ and $y_1{'}u_1{'} + y_2{'}u_2{'} = f(x)$ then $y_p = u_1y_1 + u_2y_2$ will indeed be a solution to the nonhomogeneous equation
  - Solving this system of equations, we get $u_1 = -\displaystyle\int \frac{y_2f(x)}{y_1y_2{'} - y_1{'}y_2}dx$ and $u_2 = \displaystyle\int \frac{y_1f(x)}{y_1y_2{'} - y_1{'}y_2}dx$