MATH 20D – Dr. Xiao
Section D07/D08
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Announcements

- Midterm 1 grades are released on gradescope
  - Regrade requests are now closed, but if you have any questions feel free to email me
- Second MATLAB assignment due at 1:00pm PDT on Friday
- Fourth homework set due at 11:59pm PDT on Sunday

Example Problems

1. Find the general solution to $y'' + y = \sec x$

When solving this problem, we notice that $g(x) = \sec x$ is not a function we can annihilate, meaning that we cannot use the method of undetermined coefficients. However, the corresponding homogeneous equation $y_h'' + y_h = 0$ is easy to solve so we can attempt to use variation of parameters to solve our nonhomogeneous equation.

First solve for y_h: $y_h'' + y_h = 0 \rightarrow y_h = c_1 \sin x + c_2 \cos x$

Therefore, $y_p = u_1(x)\sin x + u_2(x)\cos x$.

Using the equations for u_1, u_2 that we found in our section on variation of parameters we get:

$$u_1 = -\int \frac{y_2 g(x)}{y_1 y_2' - y_1' y_2} dx = \int \frac{\cos x \sec x}{\sin^2 x + \cos^2 x} dx = \int \cos x \sec x \, dx = \int dx = x$$

$$u_2 = \int \frac{y_1 g(x)}{y_1 y_2' - y_1' y_2} dx = \int \frac{\sin x \sec x}{-\sin^2 x - \cos^2 x} dx = -\int \sin x \sec x \, dx$$

$$= -\int \tan x \, dx = \ln|\cos x|$$

So $y_p = x \sin x + (\ln|\cos x|)\cos x$ and the general solution is:
$y = c_1 \sin x + c_2 \cos x + x \sin x + \cos x \ln|\cos x|$

2. Find the general solution to $x^2 y'' + xy' - y = \ln x$
We first start by finding the general solution to the homogeneous equation $x^2 y'' + xy' - y = 0$. This is a Cauchy-Euler equation with $x_0 = 0$, so our solutions will be of the form $y = x^r$. Solving for our characteristic equation we get: $r(r-1) + r - 1 = 0 \rightarrow r^2 - 1 = 0 \rightarrow r = \pm 1$. So $y_h = c_1 x + c_2 x^{-1}$.

Next we will use variation of parameters to solve for y_p, making use of integration by parts to solve the integrals:
$y_p = u_1 x + u_2 x^{-1}$

$$u_1 = -\int \frac{x^{-1}\ln x}{x\left(-x^{-2}\right) - x^{-1}} dx = \frac{1}{2}\int \ln x \, dx \to u = \ln x,\ v' = 1 \to u' = x^{-1}, v = x$$

$$\to \frac{1}{2}\left[x \ln x - \int (1) dx\right] = \frac{1}{2}[x \ln x - x] = \frac{x}{2}[\ln x - 1]$$

$$u_2 = \int \frac{x \ln x}{x\left(-x^{-2}\right) - x^{-1}} dx = -\frac{1}{2}\int x^2 \ln x \, dx \to u = \ln x,\ v' = x^2 \to u' = x^{-1},\ v = \frac{x^3}{3}$$

$$\to -\frac{1}{2}\left[\frac{x^3 \ln x}{3} - \int \frac{x^2}{3} dx\right] = -\frac{1}{2}\left[\frac{x^3 \ln x}{3} - \frac{x^3}{9}\right] = \frac{x^3}{6}\left[\frac{1}{3} - \ln x\right]$$

$$y_p = \frac{x}{2}[\ln x - 1] * x + \frac{x^3}{6}\left[\frac{1}{3} - \ln x\right] * x^{-1} = \frac{x^2 \ln x}{3} - \frac{4x^2}{9} = \frac{x^2}{3}\left[\ln x - \frac{4}{3}\right]$$

Therefore, the general solution is $y = c_1 x + c_2 x^{-1} + \frac{x^2}{3}\left[\ln x - \frac{4}{3}\right]$

3. Use the definition of the Laplace Transform to find $\mathscr{L}\{1\}, \mathscr{L}\{x\}, \mathscr{L}\left\{x^n\right\}$

First let's solve: $\mathscr{L}\{1\} = \int_0^\infty e^{-sx} dx = \left[-\frac{e^{-sx}}{s}\right]_0^\infty = 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}.$

Next we can try $\mathscr{L}\{x\}$ by using integration by parts:

$$\mathscr{L}\{x\} = \int_0^\infty x e^{-sx} dx \to u = x,\ v' = e^{-sx} \to u' = 1,\ v = -\frac{e^{-sx}}{s} \to \left[-\frac{x e^{-sx}}{s}\right]_0^\infty + \int_0^\infty \frac{e^{-sx}}{s} dx$$

But we need to use L'Hopital's rule to evaluate the limit in the first term of this result:

$\lim_{x \to \infty}\left(\frac{-x}{s e^{sx}}\right) = \lim_{x \to \infty} \frac{-1}{s^2 e^{sx}} = 0$, so the result is that

$$\mathscr{L}\{x\} = \int_0^\infty x e^{-sx} dx = \int_0^\infty \frac{e^{-sx}}{s} dx = \left[-\frac{e^{-sx}}{s^2}\right]_0^\infty = 0 - \left(-\frac{1}{s^2}\right) = \frac{1}{s^2}$$

Finally, let's examine $\mathscr{L}\left\{x^n\right\}$:

If we use integration by parts we will get:

$$\mathscr{L}\left\{x^n\right\} = \int_0^\infty x^n e^{-sx} dx \to u = x^n,\ v' = e^{-sx} \to u' = n x^{n-1},\ v = -\frac{e^{-sx}}{s}$$

$\rightarrow \left[-\frac{x^n e^{-sx}}{s}\right]_0^\infty + \int_0^\infty \frac{nx^{n-1}e^{-sx}}{s}dx$

By applying L'Hopital's rule like before we will get

$\mathscr{L}\left\{x^n\right\} = \int_0^\infty x^n e^{-sx}dx = \int_0^\infty \frac{nx^{n-1}e^{-sx}}{s}dx = \frac{n}{s}\int_0^\infty x^{n-1}e^{-sx}dx = \frac{n}{s}\mathscr{L}\left\{x^{n-1}\right\}.$

Successive applications of the integration by parts method will yield:

$\mathscr{L}\left\{x^n\right\} = \frac{n!}{s^n}\mathscr{L}\{1\} = \frac{n!}{s^n}*\frac{1}{s} = \frac{n!}{s^{n+1}}.$

4. Use the definition of the Laplace Transform to find $\mathscr{L}\{f(x)\}$ where $f(x) = \begin{cases} x & 0 < x < 1 \\ 0 & x > 1 \end{cases}$

$\mathscr{L}\{f(x)\} = \int_0^1 xe^{-sx}dx + \int_1^\infty 0*e^{-sx}dx = \int_0^1 xe^{-sx}dx = \left[-\frac{xe^{-sx}}{s}\right]_0^1 + \int_0^1 \frac{e^{-sx}}{s}dx$

$= -\frac{e^{-s}}{s} + \left[-\frac{e^{-sx}}{s^2}\right]_0^1 = -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} = -\frac{e^{-s}}{s}\left(1+\frac{1}{s}\right) + \frac{1}{s^2} = -\frac{e^{-s}}{s}*\frac{s+1}{s} + \frac{1}{s^2}$

$= -\frac{e^{-s}(s+1)}{s^2} + \frac{1}{s^2} = \frac{1-e^{-s}(s+1)}{s^2}$

Variation of Parameters

- Reduction of order
  - Let's say we have one solution y_1 for a second-order linear homogeneous equation in standard form $y'' + P(x)y' + Q(x) = 0$
  - Now let's assume that $y_2 = u(x)*y_1$ for some function $u(x)$
  - Taking derivatives and simplifying, we get $y_1u'' \ + \ (2y_1{}' + Py_1)u' = 0$
  - If we choose the substitution $w = u'$, then we get a separable first-order equation that we can solve
  - Ultimately, we get $u(x) = \int \frac{e^{-\int P(x)dx}}{[y_1(x)]^2}dx$
  - Example: We know that $y_1 = e^{-x}$ is a solution to the equation $y'' + 2y' + y = 0$, which has repeated roots. What is the second linearly-independent solution to this equation?
    - We get $u(x) = \int \frac{e^{-\int 2dx}}{\left[e^{-x}\right]^2}dx = \int \frac{e^{-2x}}{e^{-2x}}dx = \int dx = x$, so $y_2 = xe^{-x}$, which is what we had previously found to be true!
  - The bottom line here is that multiplying one solution by a function of x can help to solve for a second solution. *You won't be asked to use this method on a test, I am just including it here because it helps explain our rationale below (which you will need to apply).*
- So now we can develop a general method for finding the particular solution y_p for a

nonhomogeneous equation $y'' + P(x)y' + Q(x)y = g(x)$ where $y_h = c_1y_1 + c_2y_2$ is the general solution of the corresponding homogeneous equation $y_h'' + P(x)y_h' + Q(x)y_h = 0$.

- We will make a similar assumption as we did above, and assume that $y_p = u_1(x)*y_1 + u_2(x)*y_2$ for some functions $u_1,\ u_2$
- Taking the derivatives and substituting: $y_p'' + P(x)y_p' + Q(x)y_p = g(x)$, then simplifying, we get: $\frac{d}{dx}[y_1u_1' + y_2u_2'] + P[y_1u_1' + y_2u_2'] + (y_1'u_1' + y_2'u_2') = g(x)$
- If we choose $y_1u_1' + y_2u_2' = 0$ and $y_1'u_1' + y_2'u_2' = g(x)$ then $y_p = u_1y_1 + u_2y_2$ will indeed be a solution to the nonhomogeneous equation
- Solving this system of equations, we get $u_1 = -\int \frac{y_2g(x)}{y_1y_2' - y_1'y_2}dx$ and $u_2 = \int \frac{y_1g(x)}{y_1y_2' - y_1'y_2}dx$

Cauchy-Euler Equations

- Equation:
  - $a(x - x_0)^2y'' + b(x - x_0)y' + cy = 0, a \neq 0$
- Method of solution:
  - We will try to find solutions of the form $y = (x - x_0)^r$ because $y' = r(x - x_0)^{r-1}$ and $y'' = r(r-1)(x - x_0)^{r-2}$
  - Therefore our equation will become: $ar(r-1)(x - x_0)^r + br(x - x_0)^r + c(x - x_0)^r = 0$
  - Which will simplify to $ar^2 + (b - a)r + c = 0$, which we will call the *characteristic equation*
  - Use quadratic formula to solve for r and don't forget about imaginary roots
  - Three cases for our solution:
    - Two real roots: general solution is $y = c_1(x - x_0)^{r_1} + c_2(x - x_0)^{r_2}$
    - One real root:
      - Our first solution is $y_1 = c_1(x - x_0)^r$ with $r = \frac{a-b}{2a}$ but we need to find a second linearly independent solution
      - We can use reduction of order to solve for the second solution, which will have the form $y_2 = u(x)*y_1$ and $u(x) = \int \frac{e^{-\int P(x)dx}}{[y_1(x)]^2}dx$
      - To do this we first have to put our equation into standard form: $y'' + \frac{b}{a(x - x_0)}y' + \frac{c}{a(x - x_0)^2}y = 0$, for $x > x_0$
      - Then $u(x) = \int e^{-\frac{b}{a}\ln(x - x_0)}(x - x_0)^{-\frac{a-b}{a}}dx = \int (x - x_0)^{-\frac{b}{a}}(x - x_0)^{\frac{b}{a}}(x - x_0)^{-1}dx = \int \frac{dx}{x - x_0} = \ln(x - x_0)$
      - So $y_2 = c_2(x - x_0)^r \ln(x - x_0)$

    - Our general solution is $y = c_1(x - x_0)^r + c_2(x - x_0)^r \ln(x - x_0)$
  * Imaginary roots:
    - These will be of form $r = \alpha \pm \beta i$
    - We can show that $(x - x_0)^{\alpha \pm \beta i} = (x - x_0)^{\alpha} e^{\pm \beta i \ln(x - x_0)}$
    - Then we use a similar method of simplification involving Euler's identity that we used for imaginary roots of the constant-coefficient homogeneous equations
    - Our general solution is $y = (x - x_0)^{\alpha}[c_1 \cos(\beta \ln(x - x_0)) + c_2 \sin(\beta \ln(x - x_0))]$
- We can solve nonhomogeneous Cauchy-Euler equations of the form $a(x - x_0)^2 y'' + b(x - x_0)y' + cy = g(x)$ by first solving the homogeneous equation as described above to find y_h and then using variation of parameters to find y_p, for the general solution $y = y_h + y_p$

Laplace Transform: Definition

- In mathematics, a transform is an operation that takes a function as an input and outputs a different function
  - Differentiation and Integration are both transforms!
- An *integral transform* is a particular kind of transform with the format

  $$T\{f(x)\} = \int_{x_1}^{x_2} K(s, x) f(x) dx = F(s)$$

  - We call the function $K(s, x)$ the *kernel* of the transform
  - The transform takes a function of x as the input and outputs a function of s
- There are many useful integral transforms in mathematics, including the Fourier Transform, which decomposes a function into its fundamental frequency components
- We will be looking at the Laplace Transform, which is particularly useful for the study of differential equations and dynamical systems

- The kernel of the Laplace Transform is $K(s, x) = e^{-sx}$ and we integrate over the interval $[0, \infty]$
- Therefore, $\mathscr{L}\{f(x)\} = \int_0^{\infty} e^{-sx} f(x) dx = \lim_{N \to \infty} \int_0^{N} e^{-sx} f(x) dx$
- We can of course calculate this integral directly for many functions, but there are a number of useful identities that are summarized by table 7.1 in the textbook. You can find the derivations for these identities there too:

Table 7.1 Brief Table of Laplace Transforms

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$
1	$\frac{1}{s}, \quad s > 0$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\sin bt$	$\frac{b}{s^2+b^2}, \quad s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}, \quad s > 0$
$e^{at}t^n, \quad n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$

- One very useful property of integral transforms comes from the linearity property of derivatives and integrals
  - We have already seen how useful this property is for derivatives when solving nonhomogeneous equations!
  - For any integral transform the following is true: $T\{af(x) + bg(x)\} = aT\{f(x)\} + bT\{g(x)\}$