MATH 20D – Dr. Xiao
Section D07/D08
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5/04/21

Announcements

- Fifth Homework due this Sunday at 11:59pm PDT
- Midterm 2 is coming up soon!
  - May 14th
  - Expect announcements to come about the material on the test
  - I will hold a review session next week, likely Wednesday May 12th, but wait for my announcement

Example Problems

1. Solve $y'''' - 2y''' + 5y'' - 8y' + 4y = 4$ with initial conditions $y(0) = 0,\ y'(0) = 1,\ y''(0) = 3,\ y'''(0) = 1.$

We notice that this can either be solved using Laplace transforms or using our undetermined coefficients–annihilator approach we discussed a couple weeks ago. I will go through both methods of solution so you can compare them.

Laplace method:

$$\mathscr{L}\{y'''' - 2y''' + 5y'' - 8y' + 4y\} = \mathscr{L}\{4\} \tag{1}$$

$$\mathscr{L}\{y''''\} - 2\mathscr{L}\{y'''\} + 5\mathscr{L}\{y''\} - 8\mathscr{L}\{y'\} + 4\mathscr{L}\{y\} = \mathscr{L}\{4\} \tag{2}$$

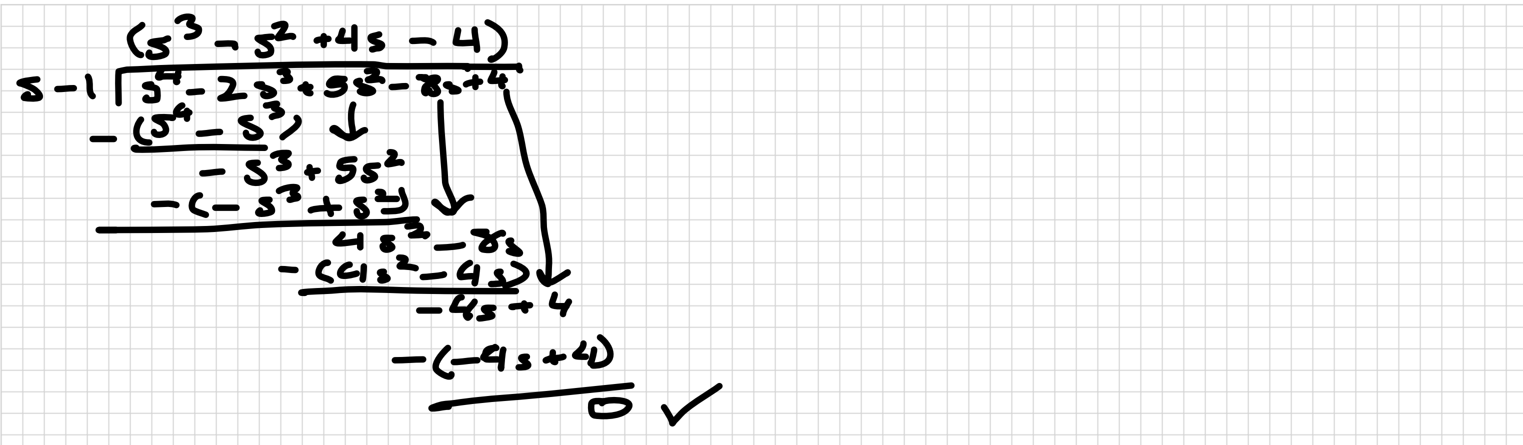
$$\begin{gathered}\left(s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)\right) - 2\left(s^3Y(s) - s^2y(0) - sy'(0) - y''(0)\right) \\ +5\left(s^2Y(s) - sy(0) - y'(0)\right) - 8(sY(s) - y(0)) + 4(Y(s)) = \frac{4}{s}\end{gathered} \tag{3}$$

$$\left[s^4 - 2s^3 + 5s^2 - 8s + 4\right]Y(s) - s^2 - s - \frac{4}{s} = 0 \tag{4}$$

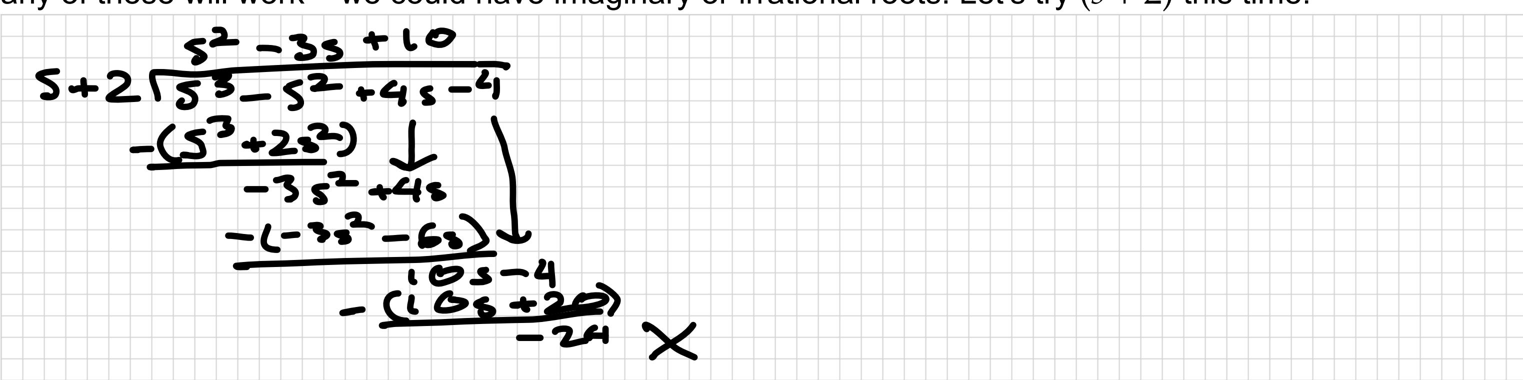
$$s\left[s^4 - 2s^3 + 5s^2 - 8s + 4\right]Y(s) = \left[s^3 + s^2 + 4\right] \tag{5}$$

$$Y(s) = \frac{s^3 + s^2 + 4}{s\left(s^4 - 2s^3 + 5s^2 - 8s + 4\right)} \tag{6}$$

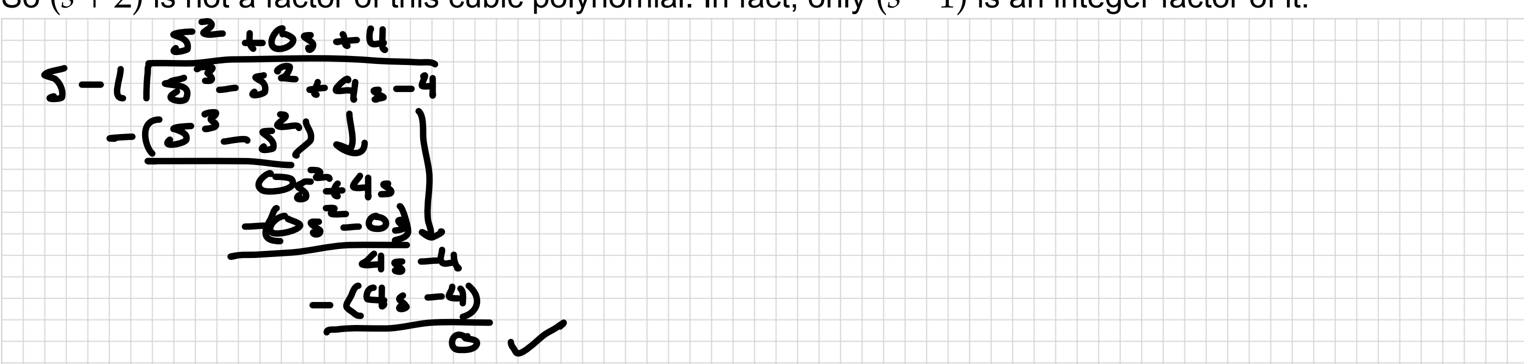
Now we need to factor the denominator. We can notice that if there are any integer roots of the quartic polynomial, it will be a factor of 4 (try multiplying the polynomial out again to confirm this is correct). Therefore we can test if any of: $(s-1),\ (s+1),\ (s-2),\ (s+2),\ (s-4),\ (s+4)$ are factors of this polynomial. Let's first try $(s-1)$:



So now we have $Y(s) = \dfrac{s^3 + s^2 + 4}{s(s-1)\left(s^3 - s^2 + 4s - 4\right)}$. Once again, we can test roots of the form $(s-1),\ (s+1),\ (s-2),\ (s+2),\ (s-4),\ (s+4)$, although we should remember that it is not guaranteed any of these will work – we could have imaginary or irrational roots! Let's try $(s+2)$ this time:



So $(s+2)$ is not a factor of this cubic polynomial. In fact, only $(s-1)$ is an integer factor of it:



$\left(s^2 + 4\right)$ has the complex roots $\pm 2i$ so we do not need to factor further. Now we can use partial fraction decomposition to aid us in finding the inverse Laplace transform:

$$Y(s) = \frac{s^3 + s^2 + 4}{s(s-1)^2\left(s^2 + 4\right)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2} + \frac{Ds + E}{s^2 + 4} \tag{7}$$

We see that 0 is a real, nonrepeated root, so we can multiply both sides of the equation by s and evaluate at $s = 0$ to find A:

$$\frac{s^3+s^2+4}{(s-1)^2\left(s^2+4\right)} = A + \left[\frac{B}{s-1} + \frac{C}{(s-1)^2} + \frac{Ds+E}{s^2+4}\right]s \tag{8}$$

$$\frac{0+4}{(0-1)^2(0+4)} = A + 0 \tag{9}$$

$$\therefore A = 1 \tag{10}$$

Now let's find a common denominator and simplify to solve for the remaining constants:

$$\frac{1(s-1)^2\left(s^2+4\right)}{s(s-1)^2\left(s^2+4\right)} + \frac{B(s-1)\left(s^2+4\right)s}{(s-1)(s-1)\left(s^2+4\right)s} + \frac{C\left(s^2+4\right)s}{(s-1)^2\left(s^2+4\right)s} + \frac{(Ds+E)(s-1)^2 s}{s^2+4(s-1)^2 s} \tag{11}$$

$$\frac{(B+D+1)s^4+(-B+C+E-2D-2)s^3+(4B-2E+D+5)s^2+(-4B+4C+E-8)s+4}{s(s-1)^2\left(s^2+4\right)} = \frac{0s^4+1s^3+1s^2+0s+4}{s(s-1)^2\left(s^2+4\right)} \tag{12}$$

$$\begin{cases} s^4: & B+D+1=0 \\ s^3: & -B+C+E-2D-2=1 \\ s^2: & 4B-2E+D+5=1 \\ s: & -4B+4C+E-8=0 \\ c: & 4=4 \end{cases} \tag{13}$$

This we can solve using algebraic methods or linear algebra, and the results are:

$$A = 1;\ B = -\frac{17}{25};\ C = \frac{6}{5};\ D = -\frac{8}{25};\ E = \frac{12}{25} \tag{14}$$

$$Y(s) = \frac{1}{s} - \frac{17}{25}\left(\frac{1}{s-1}\right) + \frac{6}{5}\left(\frac{1}{(s-1)^2}\right) + \frac{-\frac{8}{25}s + \frac{12}{25}}{s^2+4}$$

$$= \frac{1}{s} - \frac{17}{25}\left(\frac{1}{s-1}\right) + \frac{6}{5}\left(\frac{1}{(s-1)^2}\right) - \frac{8}{25}\left(\frac{s}{s^2+4}\right) + \frac{6}{25}\left(\frac{2}{s^2+4}\right) \tag{15}$$

$$y = 1 - \frac{17}{25}e^x + \frac{6}{5}xe^x - \frac{8}{25}\cos 2x + \frac{6}{25}\sin 2x \tag{16}$$

Undetermined Coefficients–Annihilator Approach method:

We first want to solve the corresponding homogeneous equation $y_h'''' - 2y_h''' + 5y_h'' - 8y_h' + 4y_h = 0$.

$$\left(D^4 - 2D^3 + 5D^2 - 8D + 4\right)y_h = 0 \tag{1}$$

$$(D-1)^2\left(D^2+4\right)y = 0 \tag{2}$$

$$y_h = c_1 e^x + c_2 x e^x + c_3 \cos 2x + c_4 \sin 2x \tag{3}$$

Note that the factorization of the differential operator is the same as the factorization of our denominator which we worked out earlier.

Next we need to find a particular solution y_p using the method of undetermined coefficients:

$$D(D-1)^2\left(D^2+4\right)y_p = D(4) = 0 \tag{4}$$

$$y_p = c_1e^x + c_2xe^x + c_3\cos 2x + c_4\sin 2x + c_5 = y_h + c_5 \tag{5}$$

$$y_p = c_5 \tag{6}$$

$$y_p{}' = 0,\ \ \ldots \tag{7}$$

$$\therefore 4c_5 = 4 \rightarrow c_5 = 1 \rightarrow y_p = 1 \tag{8}$$

So we get the general solution of our nonhomogeneous equation and can solve for the initial conditions:

$$y = c_1e^x + c_2xe^x + c_3\cos 2x + c_4\sin 2x + 1 \tag{9}$$

$$\begin{aligned} y' &= c_1e^x + c_2\left[e^x + xe^x\right] + c_3[-2\sin 2x] + c_4[2\cos 2x] \\ y'' &= c_1e^x + c_2\left[2e^x + xe^x\right] + c_3[-4\cos 2x] + c_4[-4\sin 2x] \\ y''' &= c_1e^x + c_2\left[3e^x + xe^x\right] + c_3[8\sin 2x] + c_4[-8\cos 2x] \end{aligned} \tag{10}$$

$$\begin{cases} y: & c_1 + c_3 + 1 = 0 \\ y': & c_1 + c_2 + 2c_4 = 1 \\ y'': & c_1 + 2c_2 - 4c_3 = 3 \\ y''': & c_1 + 3c_2 - 8c_4 = 1 \end{cases} \tag{11}$$

$$\therefore c_1 = -\frac{17}{25};\ c_2 = \frac{6}{5};\ c_3 = -\frac{8}{25};\ c_4 = \frac{6}{25} \tag{12}$$

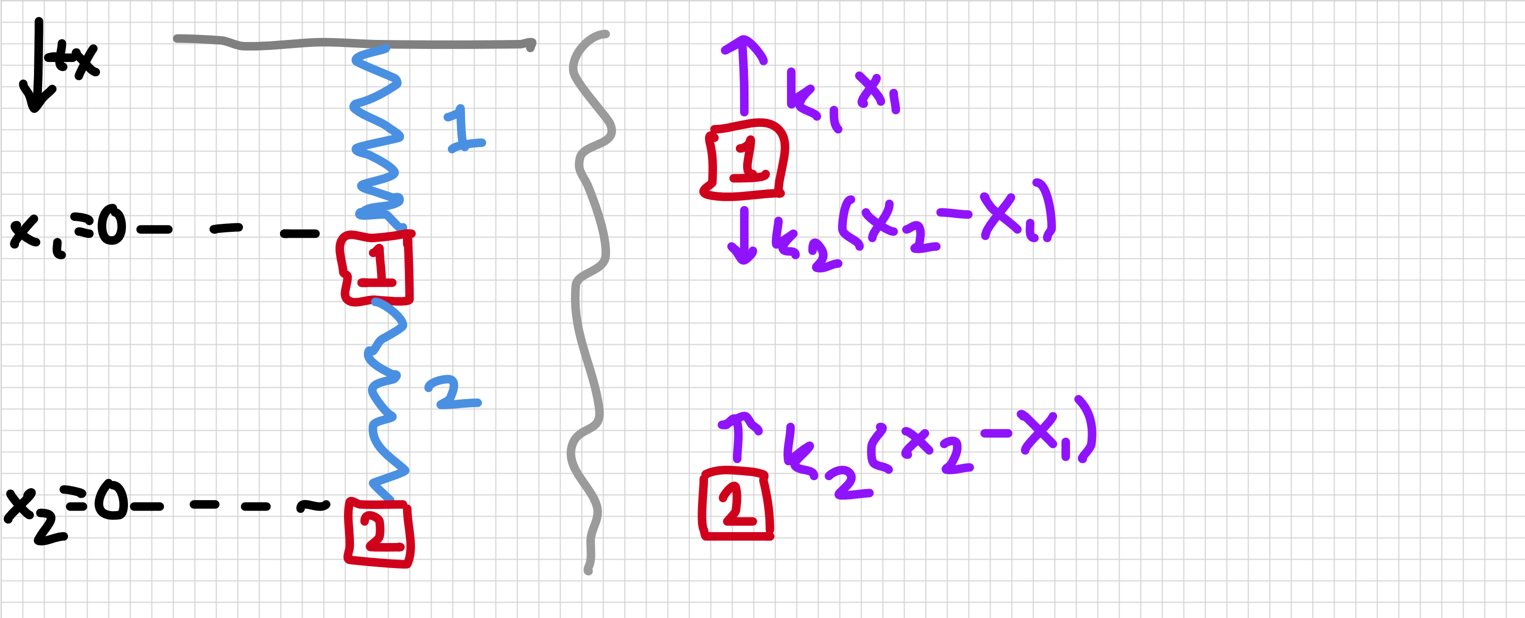
And therefore our particular solution is:

$$y = 1 - \frac{17}{25}e^x + \frac{6}{5}xe^x - \frac{8}{25}\cos 2x + \frac{6}{25}\sin 2x \tag{13}$$

Which is identical to what we found using the Laplace method!

Note: with both methods, we needed to factor a polynomial and then solve a system of linear equations for 4 constants, and therefore take about the same amount of effort. In this case, I would personally use undetermined coefficients because it requires slightly less algebra and therefore means I'm less likely to make a mistake. However, there are many problems which the Laplace method can solve that the undetermined coefficients method cannot!

2. Suppose we have two hanging masses connected with springs like this:



On the left we have a diagram depicting the springs, masses, and their equilibrium positions (note that our position axis is vertical and increases downward). On the right we show the force balance on each object based on Hooke's law $F = k*\Delta x$, where k is the spring constant and Δx is the distance that the spring is stretched. Note that we ignore gravity because that only affects the equilibrium positions of the two masses.

Mathematically, we have:

$$\sum F_1 = m_1 a_1 \rightarrow -k_1 x_1 + k_2(x_2 - x_1) = m_1 x_1''$$
$$\sum F_2 = m_2 a_2 \rightarrow -k_2(x_2 - x_1) = m_2 x_2''$$

For this problem, we will have both masses be 1kg, the spring constants are 6 N/m and 4 N/m, respectively, and both masses start at rest with displacement 20cm = 0.2m (spring 1 is stretched by 20cm and spring 2 is unstretched).

Therefore, we set $m_1 = m_2 = 1,\ k_1 = 6,\ k_2 = 4,\ x_1(0) = x_2(0) = 0.2,\ x_1'(0) = 0,\ x_2'(0) = 0.$

We get the following system of differential equations:

$$\begin{cases} x_1'' + 10x_1 - 4x_2 = 0 \\ x_2'' - 4x_1 + 4x_2 = 0 \end{cases} \tag{1}$$

Let's take the Laplace transform of both equations:

$$\begin{cases} s^2 X_1 - s x_1(0) - x_1'(0) + 10X_1 - 4X_2 = 0 \\ s^2 X_2 - s x_2(0) - x_2'(0) - 4X_1 + 4X_2 = 0 \end{cases} \tag{2}$$

$$\begin{cases} s^2 X_1 - \frac{s}{5} + 10X_1 - 4X_2 = 0 \\ s^2 X_2 - \frac{s}{5} - 4X_1 + 4X_2 = 0 \end{cases} \quad (3)$$

$$\begin{cases} \left(s^2+10\right) X_1 - 4X_2 = \frac{s}{5} \\ \left(s^2+4\right) X_2 - 4X_1 = \frac{s}{5} \end{cases} \quad (4)$$

$$\begin{cases} X_2 = -\frac{s}{20} + \frac{1}{4}\left(s^2+10\right) X_1 \\ \left(s^2+4\right) X_2 - 4X_1 = \frac{s}{5} \end{cases} \quad (5)$$

$$\left(s^2+4\right)\left[-\frac{s}{20} + \frac{1}{4}\left(s^2+10\right) X_1\right] - 4X_1 = \frac{s}{5} \quad (6)$$

$$\left[\frac{1}{4}\left(s^2+4\right)\left(s^2+10\right) - 4\right] X_1 = \frac{s}{5} + \frac{s}{20}\left(s^2+4\right) \quad (7)$$

$$\left[\frac{s^4}{4} + \frac{7s^2}{2} + 6\right] X_1 = \frac{4s + s\left(s^2+4\right)}{20} = \frac{s^3+8s}{20} \quad (8)$$

$$\left[\frac{s^4}{4} + \frac{7s^2}{2} + 6\right] X_1 = \frac{4s + s\left(s^2+4\right)}{20} = \frac{s^3+8s}{20} \quad (9)$$

$$X_1 = \frac{s^3+8s}{5s^4+70s^2+120} \quad (10)$$

$$X_2 = -\frac{s}{20} + \frac{1}{4}\left(s^2+10\right)\frac{s^3+8s}{5s^4+70s^2+120} \quad (11)$$

$$X_2 = -\frac{s\left(s^4+14s^2+24\right)}{20\left(s^4+14s^2+24\right)} + \frac{\left(s^3+8s\right)\left(s^2+10\right)}{4\left(5s^4+70s^2+120\right)} = \frac{-s^5-14s^3-24s+s^5+18s^3+80s}{20s^4+280s^2+480} = \frac{4s^3+56s}{20s^4+280s^2+480} \quad (12)$$

$$X_2 = \frac{s^3+14s}{5s^4+70s^2+120} \quad (13)$$

We see that both $X_1(s)$ and $X_2(s)$ have the same denominator:
$5s^4 + 70s^2 + 120 = 5\left(s^4 + 14s^2 + 24\right) = 5\left(s^2+12\right)\left(s^2+2\right)$

Therefore, we have to find the inverse Laplace transforms of

$X_1 = \frac{1}{5}\left(\frac{s^3+8s}{\left(s^2+12\right)\left(s^2+2\right)}\right) = \frac{1}{5}\left(\frac{As+B}{s^2+12} + \frac{Cs+D}{s^2+2}\right)$ and

$X_2 = \frac{1}{5}\left(\frac{s^3+14s}{\left(s^2+12\right)\left(s^2+2\right)}\right) = \frac{1}{5}\left(\frac{Es+F}{s^2+12} + \frac{Gs+H}{s^2+2}\right)$

For X_1, we find the constants A–D:

$$\frac{s^3+8s}{(s^2+12)(s^2+2)} = \frac{As+B}{s^2+12} + \frac{Cs+D}{s^2+2} = \frac{(As+B)(s^2+2)}{(s^2+12)(s^2+2)} + \frac{(Cs+D)(s^2+12)}{(s^2+2)(s^2+12)}$$

Therefore, $A+C=1,\ B+D=0,\ 2A+12C=8,\ 2B+12D=0 \rightarrow A=\frac{2}{5},\ B=0,\ C=\frac{3}{5},\ D=0$

For X_2, we find E–H:

$$\frac{s^3+14s}{(s^2+12)(s^2+2)} = \frac{Es+F}{s^2+12} + \frac{Gs+H}{s^2+2} = \frac{(Es+F)(s^2+2)}{(s^2+12)(s^2+2)} + \frac{(Gs+H)(s^2+12)}{(s^2+2)(s^2+12)}$$

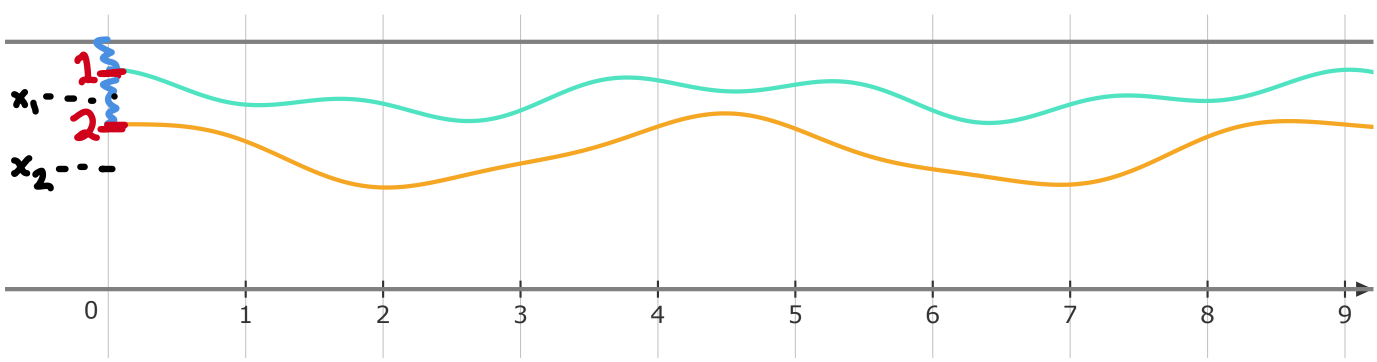
$$\rightarrow E+G=1,\ F+H=0,\ 2E+12G=14,\ 2F+12H=0 \rightarrow E=-\frac{1}{5},\ F=0,\ G=\frac{6}{5},\ H=0$$

So we will be taking the inverse Laplace transforms of:

$$\begin{cases} X_1 = \frac{2}{25}\left(\frac{s}{s^2+12}\right) + \frac{3}{25}\left(\frac{s}{s^2+2}\right) \\ X_2 = -\frac{1}{25}\left(\frac{s}{s^2+12}\right) + \frac{6}{25}\left(\frac{s}{s^2+2}\right) \end{cases} \tag{14}$$

$$x_1(t) = \frac{2}{25}\cos\left(2\sqrt{3}t\right) + \frac{3}{25}\cos\left(\sqrt{2}t\right);\ x_2(t) = -\frac{1}{25}\cos\left(2\sqrt{3}t\right) + \frac{6}{25}\cos\left(\sqrt{2}t\right) \tag{15}$$

If we assume each spring has length 40cm and the equilibrium position of the second object is 1m above the ground, we can plot the simultaneous motion of the two objects:



As we can see, both objects start 20cm above their equilibrium positions and spring 1 is compressed by 20cm. Since the motion for object 2 is coupled to object 1 through the weaker spring 2 (k=4N/m), we see that it primarily oscillates at a slower frequency, while object 1 primarily oscillates at a faster frequency associated with the stronger spring 1 (k=6N/m).

3. Find the solution to $y'' + 3xy' - 6y = 1$ with initial conditions $y(0)=0, y'(0)=0$.

We see that this is a linear second order equation, but it is neither constant-coefficient or Cauchy-Euler, so we cannot solve it through the methods we have learned previously, and must use Laplace transforms to solve.

$$\mathscr{L}\{y'' + 3xy' - 6y\} = \mathscr{L}\{1\} \tag{1}$$

$$s^2Y(s) - sy(0) - y'(0) + 3(-1)\frac{d}{ds}\mathscr{L}\{y'\} - 6Y(s) = \frac{1}{s} \tag{2}$$

$$s^2Y(s) - sy(0) - y'(0) - 3\frac{d}{ds}[sY(s) - y(0)] - 6Y(s) = \frac{1}{s} \tag{3}$$

$$s^2Y(s) - 3\frac{d}{ds}[sY(s)] - 6Y(s) = \frac{1}{s} \tag{4}$$

$$s^2Y(s) - 3(Y(s) + sY'(s)) - 6Y(s) = \frac{1}{s} \tag{5}$$

$$(-3s)Y'(s) + \left(s^2 - 9\right)Y(s) = \frac{1}{s} \tag{6}$$

$$Y'(s) + \left[-\frac{s^2 - 9}{3s}\right]Y(s) = -\frac{1}{3s^2} \tag{7}$$

We see that this is a linear first-order differential equation and can use an integrating factor to solve for $Y(s)$.

$$\mu(s) = e^{\int\left[-\frac{s^2-9}{3s}\right]ds} = e^{\int -\frac{s}{3}ds + \int \frac{3}{s}ds} = e^{-\frac{s^2}{6} + 3\ln s} = e^{-\frac{s^2}{6}}e^{3\ln s} = e^{-\frac{s^2}{6}}e^{\ln s^3} = s^3e^{-\frac{s^2}{6}}$$

$$Y(s) = \frac{\int -\frac{\mu(s)}{3s^2}ds + c}{\mu(s)} = \frac{\int\left[-\frac{s}{3}e^{-\frac{s^2}{6}}\right]ds + c}{s^3e^{-\frac{s^2}{6}}}$$

$$u = -\frac{s^2}{6} \rightarrow u' = -\frac{s}{3}$$

$$\rightarrow Y(s) = \frac{\int e^u du + c}{s^3e^{-\frac{s^2}{6}}} = \frac{e^u + c}{s^3e^{-\frac{s^2}{6}}} = \frac{1}{s^3} + \frac{c}{s^3}e^{\frac{s^2}{6}} \tag{8}$$

Since we want to find a function $y(x)$, which is continuous on $[0, \infty)$, this means that $\lim_{s\to\infty} \mathscr{L}\{y(x)\} = 0$ because $y(x)$ must be of exponential order.

Since $\lim_{s\to\infty} s^{-3}e^{\frac{s^2}{6}} = \infty$, we must choose $c = 0$ in order to get a function $Y(s)$ which has a continuous inverse Laplace transform.

Therefore:

$$y(x) = \mathscr{L}^{-1}\{Y(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{2!}{s^3}\right\} = \frac{x^2}{2} \tag{9}$$

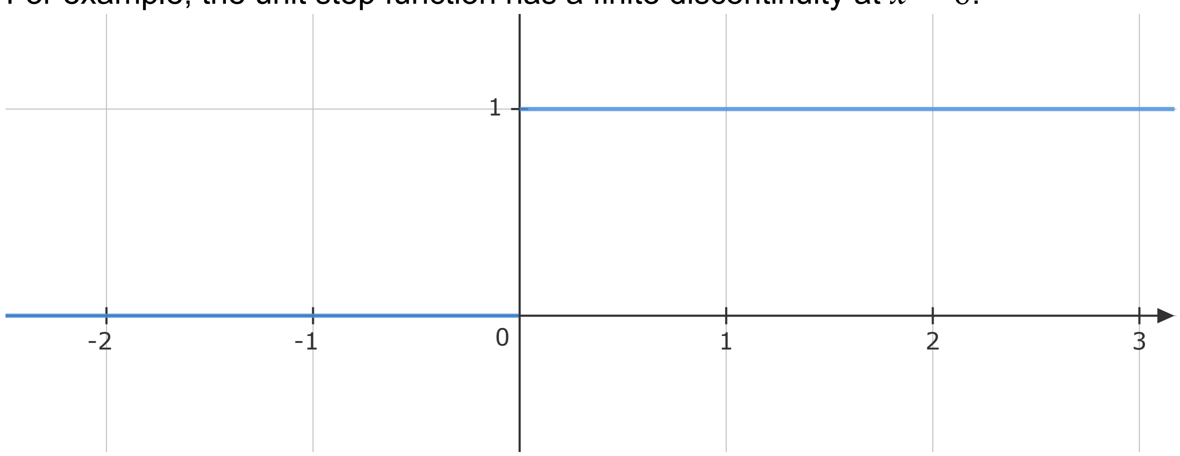
Laplace Transform: Properties

- Remember that we discussed the Linearity Property last week and reviewed table 7.1 (see last week's notes)
- Now we have a number of new properties:

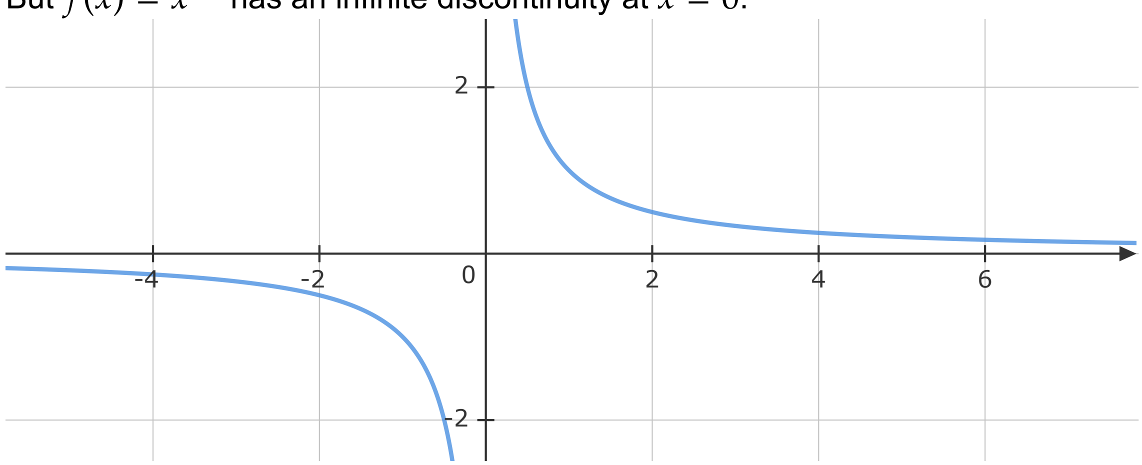
Table 7.2 Properties of Laplace Transforms

$\mathscr{L}\{f+g\} = \mathscr{L}\{f\} + \mathscr{L}\{g\}$.
$\mathscr{L}\{cf\} = c\mathscr{L}\{f\}$ for any constant c .
$\mathscr{L}\{e^{at}f(t)\}(s) = \mathscr{L}\{f\}(s-a)$.
$\mathscr{L}\{f'\}(s) = s\mathscr{L}\{f\}(s) - f(0)$.
$\mathscr{L}\{f''\}(s) = s^2\mathscr{L}\{f\}(s) - sf(0) - f'(0)$.
$\mathscr{L}\{f^{(n)}\}(s) = s^n\mathscr{L}\{f\}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$.
$\mathscr{L}\{t^n f(t)\}(s) = (-1)^n \frac{d^n}{ds^n}(\mathscr{L}\{f\}(s))$.

- Sufficient conditions for Laplace transform of a function to exist:
  - Piecewise continuous
    - Finite number points that are discontinuities on $[0, \infty)$
    - Discontinuities must be finite
      - If there is a discontinuity at $x = k$ then this is finite if:
      - $\lim_{x \to k^+} f(x)$ and $\lim_{x \to k^-} f(x)$ both exist
      - For example, the unit step function has a finite discontinuity at $x = 0$:



      - But $f(x) = x^{-1}$ has an infinite discontinuity at $x = 0$:

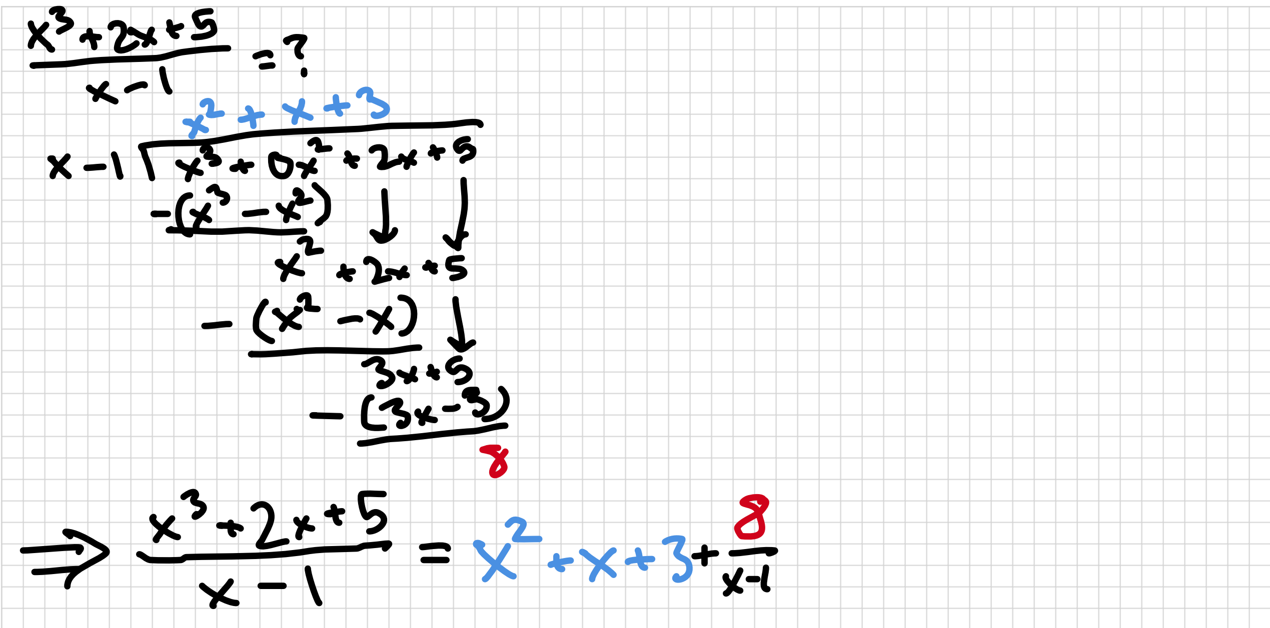


- Exponential order
    - If there exist constants c, $M > 0$, and $T > 0$, such that $|f(x)| \leqslant Me^{cx}$ for all $x > T$ then $f(x)$ is of exponential order
    - Essentially, if we can find an exponential function that "grows" faster than f, then f is of exponential order
        - This is the same as asking if $\lim_{x \to \infty} \frac{f(x)}{e^{cx}}$ exists for some value c
    - $f(x) = x^2$ is of exponential order but $f(x) = e^{x^2}$ is not
- If these two conditions are met then we are guaranteed that the Laplace transform exists for $s > c$, but there are some functions that don't meet one or both whose Laplace transforms nevertheless do exist!
    - For example, $f(x) = 2xe^{x^2}\cos\left(x^2\right)$ is not of exponential order but does have a Laplace transform!
- If both conditions are met, then $\lim_{s \to \infty} \mathscr{L}\{f(x)\} = 0$.

Laplace Transform: Calculating the Inverse Transform

- Suppose we have a function $F(s)$ that is a Laplace transform, and we want to get back the corresponding function $f(x)$ such that $\mathscr{L}\{f(x)\} = F(s)$
- If we can find a function $f(x)$ that is *continuous* over $[0, \infty)$ then that is the only such function with the Laplace transform $F(s)$
    - However there will exist an infinite number of functions which are only *piecewise continuous* which have the same Laplace transform
    - For example, in section 7.4 of the textbook, they provide the function $g(x) = \begin{cases} x & x \neq 6 \\ 0 & x = 6 \end{cases}$, which has the same Laplace transform $\frac{1}{s^2}$ as the function $f(x) = x$
- We can actually find a second integral transform $\mathscr{L}^{-1}\{F(s)\}$ which equals $f(x)$ if there exists a function $f(x)$ that is continuous over $[0, \infty)$ such that $\mathscr{L}\{f(x)\} = F(s)$.
    - We will not go into the details of this integral transform, because it is out of the scope of this class
    - However, the fact that it is another integral transform means that the inverse Laplace transform also has the linearity property!
        - $\mathscr{L}^{-1}\{aF(s) + bG(s)\} = af(x) + bg(x)$
- For this class, we will use a combination of simplification and knowledge of the properties in tables 7.1 and 7.2 to find the inverse transform
- One useful trick is *partial fraction decomposition* for the quotient of two polynomials $\frac{M(s)}{N(s)}$
    - We will often see expressions of this form when trying to find an inverse Laplace transform and would like to simplify so we can use the known identities to solve the problem
    - First, we need to make sure we have a fraction where the highest power of s in the numerator is at most one less than the highest power of s in the denominator

* If $M(s)$ is of order higher than $N(s)$, we will first use *polynomial long division* to get a sum $D(s) + \frac{R(s)}{N(s)}$, where $D(s)$ is the number of times that $N(s)$ divides $M(s)$ and $R(s)$ is the remainder
* $\frac{R(s)}{N(s)}$ is now a fraction that fits our requirement
* We do polynomial long division just like regular long division:



– Next, we need to factor the denominator as much as possible
  * We will get terms of the form $(s - r)$ for real roots and terms of the form $\left[s^2 - 2\alpha s + \left(\alpha^2 + \beta^2\right)\right]$ for imaginary roots $\alpha \pm \beta i$
  * We should end up with a fraction looking like this:

$$\frac{m_{n-1}s^{n-1} + m_{n-2}s^{n-2} \ldots m_1 s + m_0}{(s - r_1)(s - r_2) \ldots (s - r_a)\left[s^2 - 2\alpha_1 s + \left(\alpha_1^2 + \beta_1^2\right)\right] \ldots \left[s^2 - 2\alpha_b s + \left(\alpha_b^2 + \beta_b^2\right)\right]},$$

$$a + b = n$$

    · If there is a constant in the denominator after factoring, just divide that into the numerator so that we have the form above
  * We will "decompose" this fraction into the sum:

$$\frac{A}{s-r_1}+\frac{B}{s-r_2}\dots\frac{Cs+D}{s^2-2\alpha_1 s+\left(\alpha_1^2+\beta_1^2\right)}+\frac{Es+F}{s^2-2\alpha_2 s+\left(\alpha_2^2+\beta_2^2\right)}\dots$$

  * Note: if there are repeated roots, then we will have to decompose like this:
    - $\frac{M(s)}{(s-r)^n}=\frac{A}{s-r}+\frac{B}{(s-r)^2}+\frac{C}{(s-r)^3}\dots\frac{D}{(s-r)^n}$
    - $\frac{M(s)}{\left[s^2-2\alpha s+\left(\alpha^2+\beta^2\right)\right]^n}=\frac{As+B}{s^2-2\alpha s+\left(\alpha^2+\beta^2\right)}+\frac{Cs+D}{\left[s^2-2\alpha s+\left(\alpha^2+\beta^2\right)\right]^2}\dots$
  * We can solve for all the constants on non-repeated real root terms in the following way:
    - Multiply both sides of the equation by the denominator
    - Set $s=r$
    - We should get $A=\frac{M(r)}{(r-r_2)\dots}$, where $N(s)=(s-r_1)(s-r_2)\dots$
  * For repeated roots or imaginary roots, we will then solve a system of equations for the remaining constants
- Once we have completed our decomposition, we can take advantage of the linearity property of the inverse Laplace transform and use the identities in table 7.1 to find $f(x)$.
- This method might sound tedious. It really can be. But this technique is very useful for fractions where $N(s)$ is a quadratic, cubic, or quartic, and doesn't take too long in these cases.

Laplace Transform: Solving Initial Value Problems

- Now that we know how to calculate inverse Laplace transforms, we can start to solve differential equations!
- Recall that table 7.2 shows identities for the Laplace transforms of derivatives
  - These equations require us to know the initial conditions $f(0)$, $f'(0)$, etc
- If we take the Laplace transform of both sides of a differential equation, then we get an *algebraic* equation to solve for $F(s)$
- We should end up with $F(s)=\frac{M(s)}{N(s)}$
- Now we just take the inverse Laplace transform of both sides:
- $f(x)=\mathscr{L}^{-1}\{F(s)\}=\mathscr{L}^{-1}\left\{\frac{M(s)}{N(s)}\right\}$
- By solving for the inverse Laplace transform, we end up getting the particular solution to the original differential equation based on our initial conditions!