MATH 20D – Dr. Xiao
Section D07/D08
Ariel (Ari) Schreiman
5/11/21

Announcements

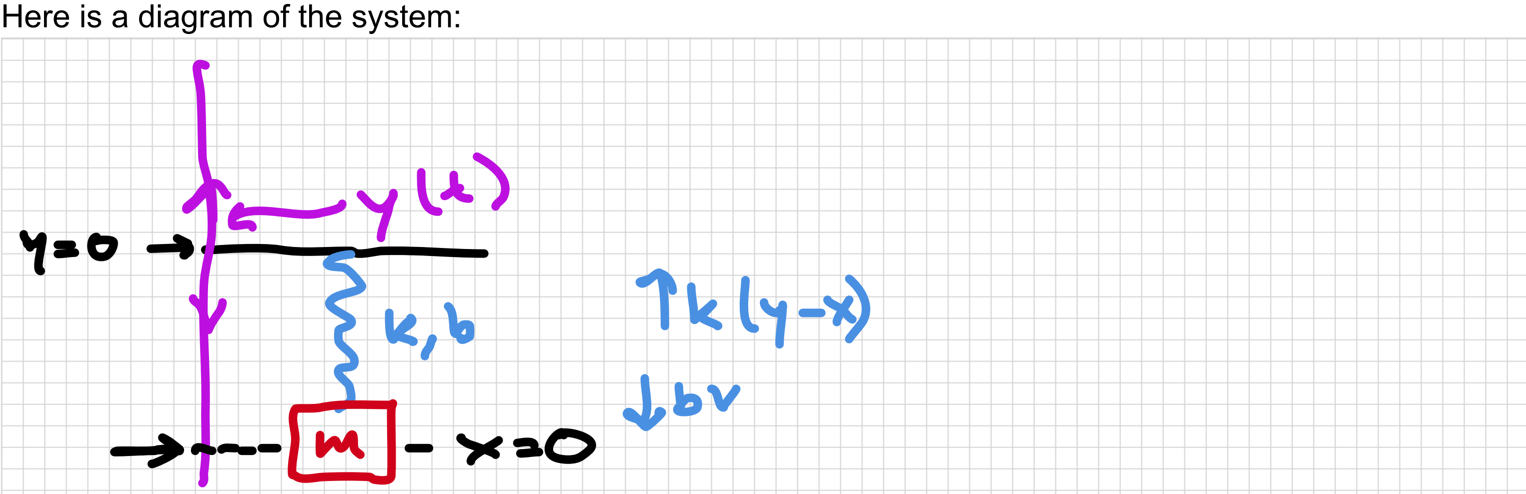
- Sixth Homework due this Sunday at 11:59pm PDT
- Third MATLAB assignment due on Friday at 11:59pm PDT
- Midterm 2 is this Friday at either 11am PDT or 6pm PDT
  - Everything we have learned up to 7.6 (no convolutions)
  - Review session on Wednesday 6pm PDT
    - https://ucsd.zoom.us/j/92402842832
- Problem 5 from last week's homework
  - Why can't we remove the c?
  - Actually, we are guaranteed a unique solution only for values of t>0 since we have the function t multiplied by y'', and we only get unique solutions when the second derivative term is nonzero.
  - So really, the problem is with our initial conditions being defined at t=0 instead of say t=1.
  - In fact, this problem is totally solvable and the general solution is: $y = c_1(t\cos t - \sin t) + c_2(t\sin t + \cos t)$, where we can see that if we use t=1 as our initial condition, we will have a unique solution

Example Problems

Control of spring hanging from platform - m = 1, b=2, k = 8, x(0)=0, x'(0)=0

We have a mass on a spring (with friction) that is hanging from a platform. We get to control the position of the platform at any time through the function y(t) and we want to see what happens to x(t).

Here is a diagram of the system:



Mathematically, we represent this by $\sum F = ma \rightarrow k(y(t) - x(t)) - bx'(t) = mx''$ rearranging we get:

$mx'' + bx' + kx = ky$. We will set $f(t) = ky(t)$ and call that the input function. This is what we get to

control.

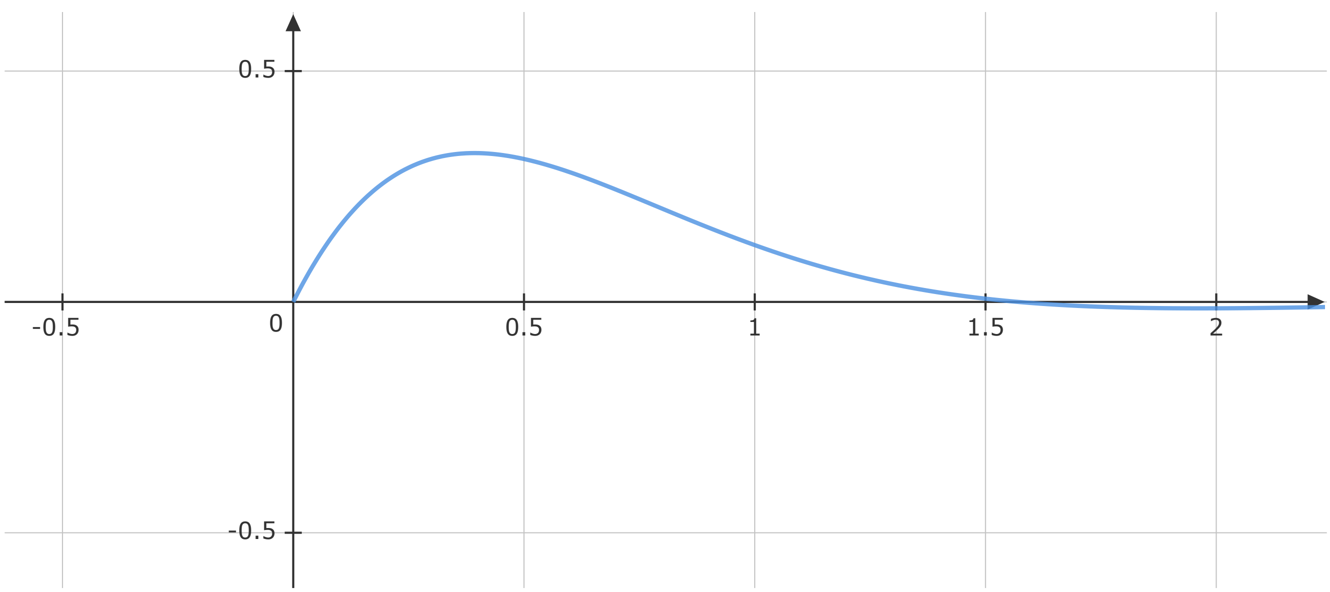
Substituting in our known values we get $x'' + 2x' + 8x = f(t)$. We want to see what happens to x(t) depending on our input, but first we will take the Laplace transform of both sides to help us understand the system better.

$$\mathscr{L}\{x'' + 2x' + 8x\} = \mathscr{L}\{f(t)\} \rightarrow s^2X(s) + 2sX(s) + 8X(s) = F(s) \rightarrow X(s) = \frac{F(s)}{(s+2)^2+4}$$

Interestingly, X(s) is the product of two functions, F(s), the input function, and $\frac{1}{(s+2)^2+4}$, which we call the transfer function because it explains how the system reacts to the input.

1. Impulse response

We jostle the platform a little bit at t=0 using the function $f(t) = 2\delta(t)$. $F(s) = 2$ and therefore $X(s) = \frac{2}{(s+2)^2+4}$. So $x(t) = e^{-2t} \sin 2t$.

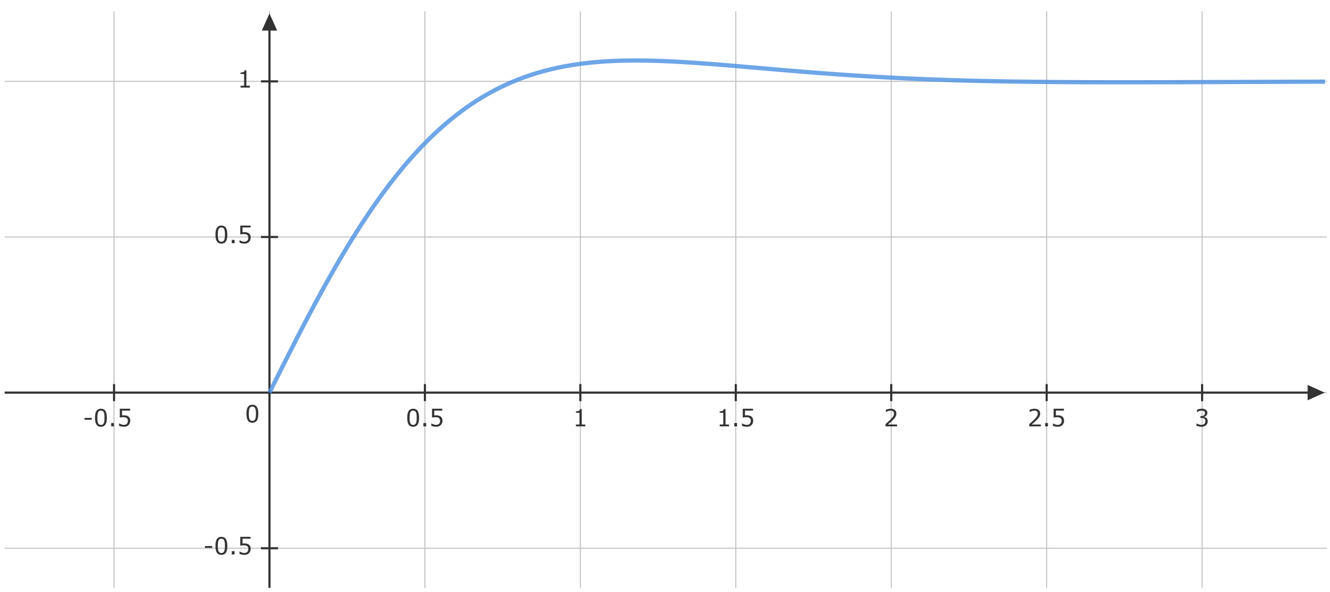


2. Step response

We move the platform up a small amount at t=0 using the function $f(t) = 8\mathscr{U}\{t\}$. $F(s) = \dfrac{8}{s}$ and therefore

$X(s) = \dfrac{8}{s\left[(s+2)^2+4\right]} = \dfrac{A}{s} + \dfrac{B(s+2)+C}{(s+2)^2+4}$. We can determine that $A = 1,\ B = -1,\ C = 0$. So

$X(s) = \dfrac{1}{s} - \dfrac{s+2}{(s+2)^2+4}$ and $x(t) = 1 - e^{-2t}\cos 2t$.



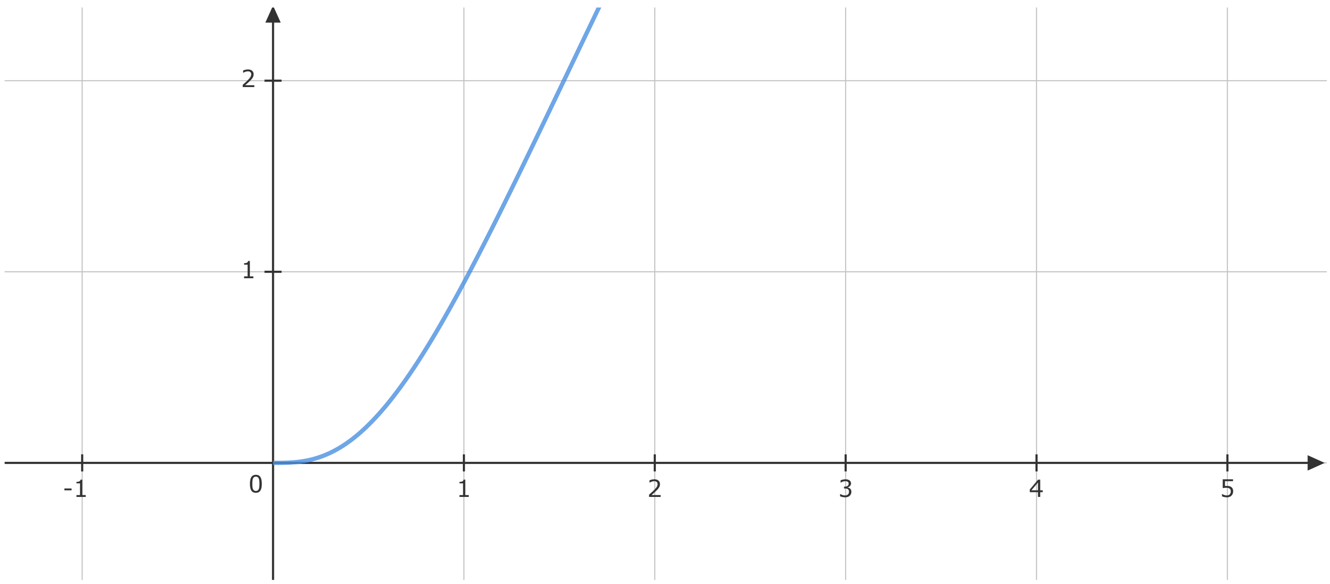
3. Ramp response

Now we will be moving the platform up continuously using the function $f(t) = 8t\mathscr{U}\{t\}$. $F(s) = \dfrac{8}{s^2}$ and

therefore $X(s) = \dfrac{8}{s^2\left[(s+2)^2+4\right]}$. Rather than using partial fraction composition this time, let's use convolutions to solve this problem (integral by parts not shown):

$x(t) = (8t) \circledast \left(e^{-2t}\sin 2t\right) = \displaystyle\int_0^t 8(t-\tau)e^{-2\tau}\sin(2\tau)\,d\tau =$

$\left[e^{-2\tau}(2(\tau-t)\sin(2\tau) + (-2t+2\tau+1)\cos(2\tau))\right]_0^t = e^{-2t}\cos 2t - [-2t+1] = (2t-1) + e^{-2t}\cos 2t$



4. PID controller response

Now, instead of just choosing any function for the input, we will instead use a controller that decides how to move the platform based on the error between the current location of the object and a desired value. We define $\epsilon(t) = g(t) - x(t)$, where $g(t)$ is the desired location of the object.

We will use a control scheme that looks at the current error (proportional term), the sum of the past errors (integral term), and the rate of change of the error (derivative term) to get the object as close to the setpoint as quickly as possible with a low amount of offshoot and with a small steady-state error.

Therefore,

$$f(t) = k_p\epsilon(t) + k_i\int_0^t \epsilon(\tau)d\tau + k_d\epsilon'(t)$$

$$= \left(k_p g(t) + k_i\int_0^t g(\tau)d\tau + k_d g'(t)\right) - \left(k_p x(t) + k_i\int_0^t x(\tau)d\tau + k_d x'(t)\right)$$

and $F(s) = \left(k_p + \frac{k_i}{s} + sk_d\right)G(s) - \left(k_p + \frac{k_i}{s} + sk_d\right)X(s)$

Manipulating our equation $X(s) = \frac{F(s)}{(s+2)^2+4}$ a bit we get:

$X(s) = \frac{\left(sk_p + k_i + s^2k_d\right)G(s)}{\left(s^3 + (2+k_d)s^2 + (8+k_p)s + k_i\right)}$. Let's decide that we want to move the object to the new position $x = 1$. That means that $g(t) = \mathscr{U}\{t\} \to G(s) = \frac{1}{s}$.

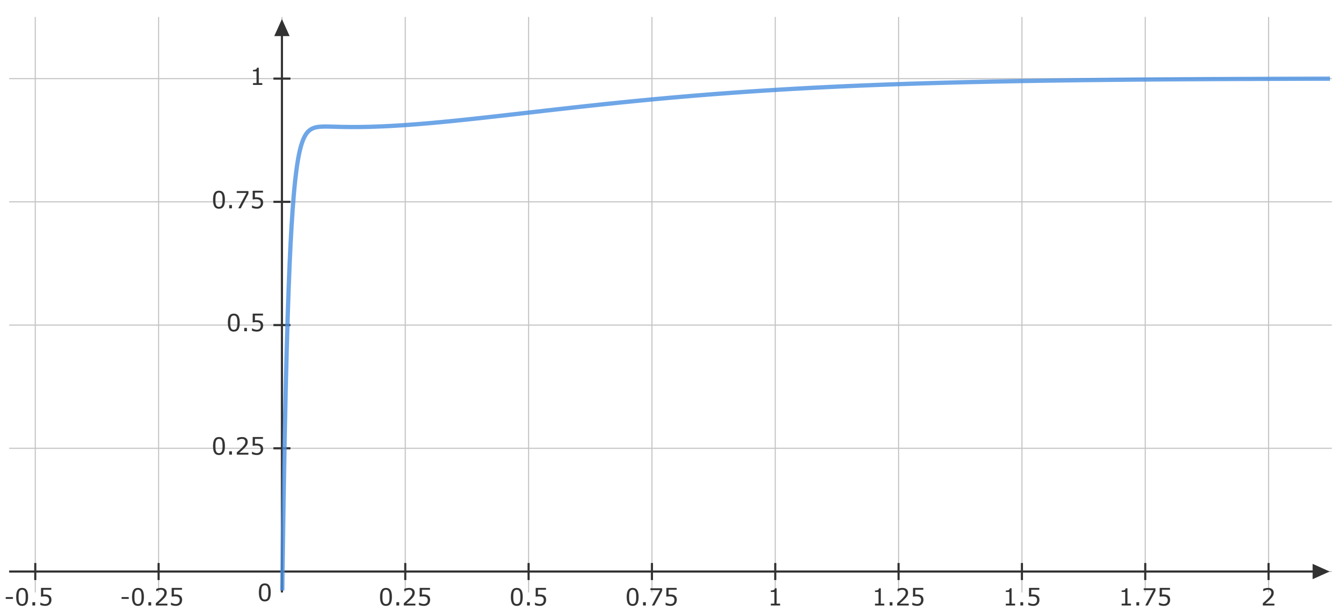
So overall, $X(s) = \frac{s^2k_d + sk_p + k_i}{s\left(s^3 + (2+k_d)s^2 + (8+k_p)s + k_i\right)}$, and after some tuning we find that $k_p = 193.5625,\ k_i = 125,\ k_d = 80.5$ work well and makes our equation relatively easy to deal with.

Therefore (you can verify the factorization in the denominator):

$$X(s) = \frac{80.5s^2 + 193.5625s + 125}{s(s+80)\left(s+\frac{5}{4}\right)^2} = \frac{A}{s} + \frac{B}{s+80} + \frac{C}{s+\frac{5}{4}} + \frac{D}{\left(s+\frac{5}{4}\right)^2}$$

$$\approx \frac{1}{s} - \frac{1.01}{s+80} + \frac{0.01}{s+\frac{5}{4}} - \frac{0.09}{\left(s+\frac{5}{4}\right)^2}$$

And so $x(t) = 1 - 1.01e^{-80t} + 0.01e^{-1.25t} - 0.09te^{-1.25t}$



Compare this to our uncontrolled step response and see how much faster we approach the setpoint (and we could do even better if I optimized the PID values more)!

Laplace Transform: Step Functions

- The step function is defined as $\mathscr{U}(x-a) = \begin{cases} 0 & x < a \\ 1 & x > a \end{cases}$
- Multiplying a step function by another function "switches on" that function at a
- We can also create a "window" by first switching on and then switching off the same function:
  $f(x)[\mathscr{U}(x-a) - \mathscr{U}(x-b)] = \begin{cases} 0 & x < a \\ f(x) & a < x < b \\ 0 & x > b \end{cases}$
  - Remember that this is just the combination of two step functions!
- Step functions are useful for modeling almost-instantaneous physical transitions like turning on a switch, collisions between objects, a part of a structure suddenly breaking, etc
- Laplace transforms can help us solve differential equations that model a physical system which experiences such events.
- New identities:
  - $\mathscr{L}\{\mathscr{U}(x-a)\} = \dfrac{e^{-as}}{s}$
  - $\mathscr{L}\{f(x)\mathscr{U}(x-a)\} = e^{-as}\mathscr{L}\{f(x+a)\}$
  - $\mathscr{L}^{-1}\left\{e^{-as}F(s)\right\} = f(x-a)\mathscr{U}(x-a)$
- One thing we might wonder, is if a unit step function is the result of a switch turning on, what was the cause, or derivative?
  - We are looking for a function where $\displaystyle\int_0^x f(t)dt = \mathscr{U}(x-a)$
  - Everywhere except $x = a$, $\dfrac{d}{dx}\mathscr{U} = 0$, but at $x = a$ this function will have an undefined (large) value such that $\displaystyle\int_{a-\Delta x}^{a+\Delta x} f(t)dt = 1$ for any chosen value of Δx, however small
  - This function is called the Dirac Delta Function, or the Unit Impulse and represented by $\delta(x-a) = \begin{cases} \infty & x = a \\ 0 & x \neq a \end{cases}$
  - And it turns out $\mathscr{L}\{\delta(x-a)\} = e^{-as}$
  - Though you aren't being tested on this function in class, both the unit step and the unit impulse functions are useful for studying physical systems

Laplace Transform: Convolutions

- A convolution $f \circledast g = \displaystyle\int_0^t f(t-\tau)g(\tau)d\tau$ is a special kind of integral transform with $f(t-\tau)$ as the kernel

- Therefore the linearity property we saw in both the Laplace and Inverse Laplace transforms still applies here!
- In addition, we can actually prove that convolutions are commutative: $f \circledast g = g \circledast f$

- New identities:
  - $\mathscr{L}\{f \circledast g\} = F(s)G(s)$
  - $\mathscr{L}^{-1}\{F(s)G(s)\} = f \circledast g$