MATH 20D – Dr. Xiao
Section D07/D08
Ariel (Ari) Schreiman
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Announcements

- Well done on the second midterm!
  - Grades should be finished by Friday
- Homework 7 due Sunday at 11:59pm PDT

Example Problems

$\left(x^2-x-1\right)y''-y'-y=0;\ y(0)=0,\ y'(0)=1$

Since this is a linear equation that is neither constant-coefficient nor Cauchy-Euler, we can either try solving this with Laplace transforms or with a series solution. However, because of the x^2y'' term, if we tried using Laplace transforms we would end up with a new second-order differential equation which is no easier to solve than the original. Therefore, we must use an infinite series to solve this problem.

First, we notice that because our initial conditions are at $x_0=0$, we would like our series to be centered there. If we put our equation in standard form, we see that $P(x), Q(x), g(x)$ are all analytic there, so we can use a solution of the form $y=\sum_{k=0}^{\infty} a_k x^k$. Taking derivatives, we get $y'=\sum_{k=1}^{\infty} a_k(k)x^{k-1}$ and $y''=\sum_{k=2}^{\infty} a_k(k)(k-1)x^{k-2}$.

Plugging into the differential equation and simplifying:

$$\left(x^2-x-1\right)\sum_{k=2}^{\infty} a_k(k)(k-1)x^{k-2}-\sum_{k=1}^{\infty} a_k(k)x^{k-1}-\sum_{k=0}^{\infty} a_k x^k=0$$

$$\sum_{n=2}^{\infty} a_n(n)(n-1)x^n-\sum_{n=2}^{\infty} a_{n+1}(n+1)(n)x^n-(2a_2x)-\sum_{n=2}^{\infty} a_{n+2}(n+2)(n+1)x^n-(2a_2+6a_3x)$$
$$-\sum_{n=2}^{\infty} a_{n+1}(n+1)x^n-(a_1+2a_2x)-\sum_{n=2}^{\infty} a_n x^n-(a_0+a_1x)=0$$

$$\sum_{n=2}^{\infty} x^n\left[a_n(n)(n-1)-a_{n+1}(n+1)(n)-a_{n+2}(n+2)(n+1)-a_{n+1}(n+1)x^n-a_n\right]-(2a_2x)-(2a_2+6a_3x)$$
$$-(a_1+2a_2x)-(a_0+a_1x)=0$$

$$\sum_{n=2}^{\infty} x^n\left[a_n\left(n^2-n-1\right)-a_{n+1}\left(n^2+1\right)-a_{n+2}\left(n^2+3n+2\right)\right]$$

$-(2a_2 + a_1 + a_0) - x(a_1 + 4a_2 + 6a_3) = 0$
Notice how we change the indices in our summations so that we can combine them all into one summation.

Now, the coefficient for each power of x must be equal to zero for the equation to be valid:
$2a_2 + a_1 + a_0 = 0$

$a_1 + 4a_2 + 6a_3 = 0$
$a_n\left(n^2 - n - 1\right) - a_{n+1}\left(n^2 + 1\right) - a_{n+2}\left(n^2 + 3n + 2\right) = 0,\ n \geqslant 2$

Referring back to our original equation and our initial conditions, we know that $a_0 = 0$ and $a_1 = 1$. Using our first two equations above, we get $a_2 = -\frac{1}{2},\ a_3 = \frac{1}{6}$. Next, by manipulating the third equation above we can find an equation for all other constants a_k:

$$a_k = \frac{a_{k-2}\left(k^2 - 5k + 5\right) - a_{k-1}\left(k^2 - 4k + 5\right)}{k^2 - k},\ k \geqslant 4$$

$$\rightarrow a_k = a_{k-2}\left(1 - \frac{5}{k} + \frac{1}{k-1}\right) - a_{k-1}\left(1 - \frac{5}{k} + \frac{2}{k-1}\right),\ k \geqslant 4$$

$$\therefore\ a_4 = -\frac{1}{9},\ a_5 = \frac{7}{72},\ a_6 = -\frac{23}{240},\ \ldots$$

Now, if we want to know where our series solution converges, we can use the ratio test:

$$\lim_{k\to\infty}\left|\frac{a_{k+1}x^{k+1}}{a_k x^k}\right| < 1 \tag{1}$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}x^{k+1}}{a_k x^k}\right| = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}x\right| \tag{2}$$

$$= \left[\lim_{k\to\infty}\left|\frac{a_{k-1}\left(1-\frac{5}{k}+\frac{1}{k-1}\right) - a_k\left(1-\frac{5}{k}+\frac{2}{k-1}\right)}{a_{k-2}\left(1-\frac{5}{k}+\frac{1}{k-1}\right) - a_{k-1}\left(1-\frac{5}{k}+\frac{2}{k-1}\right)}\right|\right]|x| \tag{3}$$

$$= \left[\lim_{k\to\infty}\left|\frac{a_{k-1}\left(1-\frac{5}{k}+\frac{1}{k-1}\right) - \left[a_{k-2}\left(1-\frac{5}{k}+\frac{1}{k-1}\right) - a_{k-1}\left(1-\frac{5}{k}+\frac{2}{k-1}\right)\right]\left(1-\frac{5}{k}+\frac{2}{k-1}\right)}{a_{k-2}\left(1-\frac{5}{k}+\frac{1}{k-1}\right) - a_{k-1}\left(1-\frac{5}{k}+\frac{2}{k-1}\right)}\right|\right]|x| \tag{4}$$

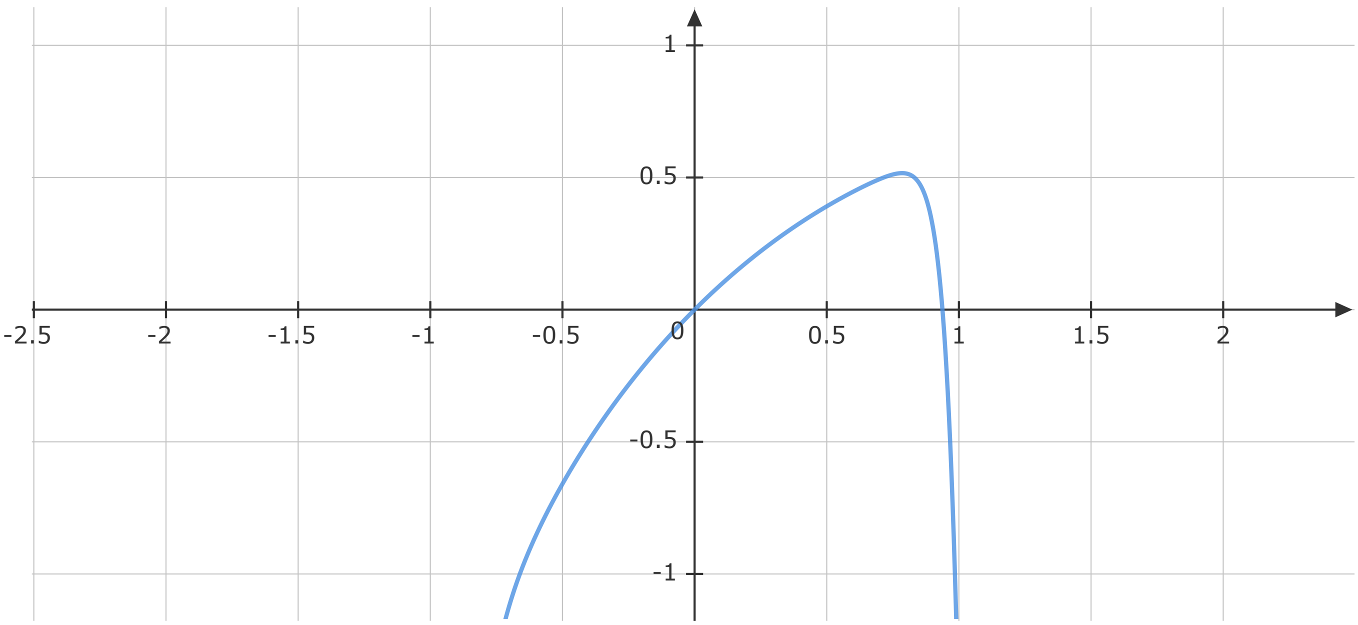
$$= \left[\lim_{k\to\infty}\left|\frac{a_{k-1} - [a_{k-2} - a_{k-1}]}{a_{k-2} - a_{k-1}}\right|\right]|x| = \left[\lim_{k\to\infty}\left|-\frac{a_{k-1}}{a_{k-1} - a_{k-2}} - 1\right|\right]|x| \tag{5}$$

$$= \left[\lim_{k \to \infty} \left| \frac{a_{k-1}}{a_{k-1} - a_{k-2}} + 1 \right| \right] |x| \leqslant 2|x| \tag{6}$$

Note that all fractions with k in the denominator approach zero as k approaches infinity. And note that because successive terms alternate in sign, $|a_{k-1} - a_{k-2}| \geqslant |a_{k-1}|$, so $0 < \dfrac{a_{k-1}}{a_{k-1} - a_{k-2}} \leqslant 1$.

Therefore, we can confirm that our series converges on at least $-\dfrac{1}{2} < x < \dfrac{1}{2}$.

Here is a graph of a 20-term approximation of the infinite series, and we do see that the graph appears to diverge for values of x close to +1 and -1:



Laplace Transform: Dirac Delta Function

- Turns out I was wrong last week, you will need to know this function for this class!
- See last week's notes for a description of this function and an example

Series Solutions to Linear Differential Equations

- Sometimes, we can solve a differential equation in terms of an infinite series called a power series

  $$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

- Motivation – Taylor's Theorem
  - For a function which is *analytic* at x_0, there exists a unique power series expansion of the function at this point
    - A function is analytic at a point when its value is defined, and its derivative is continuous at this point

* At this point the function actually has infinitely many derivatives which are continuous!
  - The proof of this remarkable theorem is outside the scope of this class
- The expansion is based on the idea that if, at a certain point, two functions are analytic and all of their derivatives have equal value, then the two are actually the same function at that point
- Therefore, we just need to find a power series where each derivative is equal to the derivative of the function:

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k \tag{7}$$

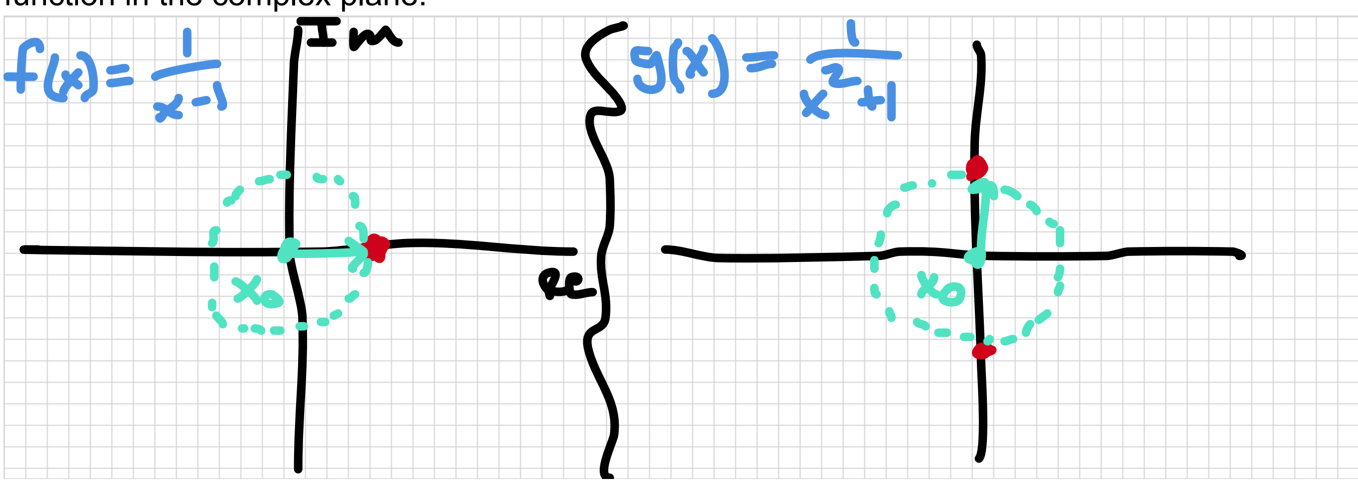
$$f'(x) = \sum_{k=1}^{\infty} a_k k (x - x_0)^{k-1} \tag{8}$$

$$f^{(n)}(x) = \sum_{k=n}^{\infty} a_k (k)(k-1)\dots(k-n+1)(x - x_0)^{k-n} \tag{9}$$

$$f^{(n)}(x_0) = a_n (n)(n-1)\dots(n-n+1) = a_n n! \tag{10}$$

$$\therefore a_n = \frac{f^{(n)}(x_0)}{n!} \tag{11}$$

- Since all terms of the power series except for the first are zero at x_0, it is easy to find the value of each coefficient
- To summarize, we can represent any function that is analytic at x_0 by the power series: $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$ which is called a Taylor series
- We can determine the radius of convergence by finding the closest *singular value* of the function in the complex plane:



- Some useful Taylor series expansions can be found at: https://en.wikipedia.org/wiki/Taylor_series
  * Note that the identities listed are referred to as Maclaurin series because they are centered at $x_0 = 0$

• Since Taylor's Theorem tells us that we can find a series expansion for every analytic function based solely on its derivatives, this suggests that power series are a potentially useful tool for solving differential equations!

- Method of solution:
  - We start with a linear second order differential equation in the standard form: $y'' + P(x)y' + Q(x)y = g(x)$
  - Now, we choose a point x_0 for our expansion where $P(x_0), Q(x_0), g(x_0)$ are all analytic
  - We assume there is a solution of the form $y = \sum_{k=0}^{\infty} a_k (x - x_0)^k$
  - We take the necessary derivatives of y and substitute into our equation
  - Replace $P(x), Q(x), g(x)$ by their Taylor series expansions around x_0
  - Simplify the equation to get $\sum_{n=0}^{\infty} c_n (x - x_0)^n = 0$
  - Therefore, each constant c_n should equal to zero and we can use this to solve for each value a_k in our original power series
    * We will need to use our initial conditions to help us find numerical values for all the constants
  - Now the last thing we must do is determine the radius of convergence for this series:
    * Since we don't necessarily know an elementary function whose Taylor series is equal to our power series result, we can't figure this out based on singular points
    * Instead, we must use a convergence test:
      · https://en.wikipedia.org/wiki/Convergence_tests
      · The simplest is the ratio test, but there are many more