

**Announcements**

- Midterm grades released, request any regrades as soon as possible
- Homework 8 due this Sunday at 11:59pm PDT
- Fourth MATLAB assignment due on Friday at 11:59pm PDT
- Revisiting the ratio test:

– Classical ratio test:  $\sum_{n=0}^{\infty} a_n \rightarrow \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$

– Manipulated form of ratio test we saw in class:  $\sum_{n=0}^{\infty} a_n x^n \rightarrow |x| < \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = r$

- Next week will be the last discussion sections for this quarter, we will go over chapter 9
  - I will also discuss details for a final exam review session then
- TA Evaluations!
  - Separate from Course/Professor evaluation (CAPE)
  - Due Monday June 7th at 11:59pm PDT

**Example Problems**

1. Airy Function (encountered in physics, electronics, and optics):

$$y'' + xy = 0$$

Since there are no singular values for  $x$ , no matter what value of  $x_0$  we choose, it will converge for  $x \in \mathbb{R}$ .

So we will choose  $x_0 = 0$  for our power series.

$$y = \sum_{k=0}^{\infty} a_k x^k \rightarrow y'' = \sum_{k=2}^{\infty} a_k (k)(k-1) x^{k-2} \tag{1}$$

$$\sum_{k=2}^{\infty} a_k (k)(k-1) x^{k-2} + x \sum_{k=0}^{\infty} a_k x^k = 0 \tag{2}$$

$$\sum_{k=2}^{\infty} a_k (k)(k-1) x^{k-2} + \sum_{k=0}^{\infty} a_k x^{k+1} = 0 \tag{3}$$

$$2a_2 + \sum_{n=1}^{\infty} a_{n+2} (n+2)(n+1) x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0 \tag{4}$$

$$2a_2 + \sum_{n=1}^{\infty} [a_{n+2} (n+2)(n+1) + a_{n-1}] x^n = 0 \tag{5}$$

Therefore, we get the system of equations:

$$a_2 = 0$$

$$a_{n+2}(n+2)(n+1) + a_{n-1} = 0, \quad n \geq 1$$

And from our original equation, we get:

$$a_0 = y(0), \quad a_1 = y'(0)$$

Using substitution to solve for each successive coefficient, we find that, for all integers  $m \geq 0$ :

$$\begin{cases} a_n = 0 & n = 3m + 2 \\ a_n = (-1)^m \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot \dots \cdot (3m-1)(3m)} & n = 3m \\ a_n = (-1)^m \frac{a_1}{3 \cdot 4 \cdot 4 \cdot 5 \cdot \dots \cdot (3m)(3m+1)} & n = 3m + 1 \end{cases}$$

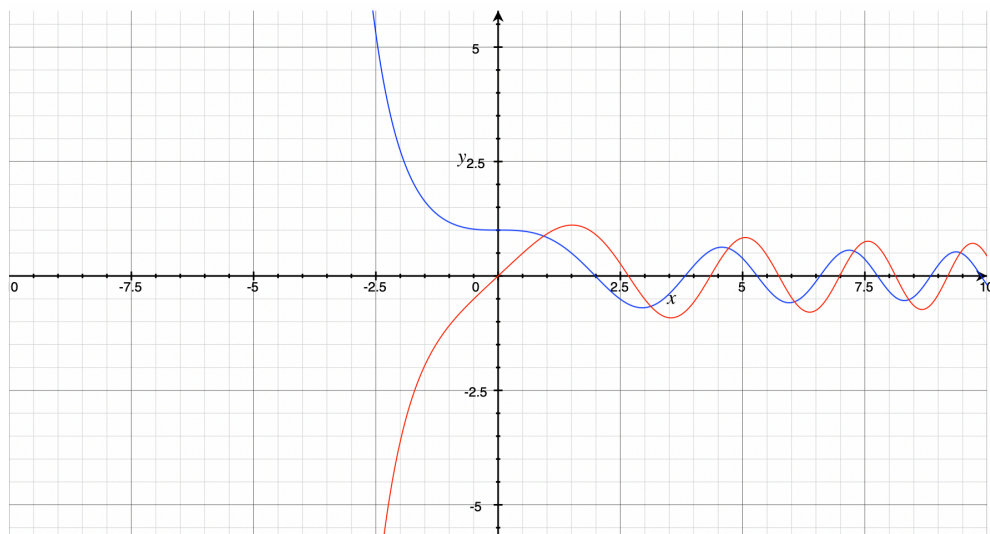
Therefore,

$$\begin{aligned} y &= a_0 \left[ 1 - \frac{1}{6}x^3 + \frac{1}{180}x^6 \dots \right] + a_1 \left[ x - \frac{1}{12}x^4 + \frac{1}{504}x^7 \dots \right] \\ &= a_0 \left[ 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\prod_{k=0}^{n-1} (3k+1)}{(3n)!} x^{3n} \right] + a_1 \left[ x + \sum_{n=1}^{\infty} (-1)^n \frac{\prod_{k=0}^{n-1} (3k+2)}{(3n+1)!} x^{3n+1} \right] \end{aligned}$$

We notice that this matches our expectations for a second order linear homogeneous equation having two solutions of the form  $y = c_1 y_1 + c_2 y_2$ , where  $y_1, y_2$  are linearly independent.

If we use the ratio test for either of these two solutions, we find that the radius of convergence is indeed infinite, as expected.

We can plot an approximation of each of these two solutions (100 terms):



Blue is the first solution and red is the second solution.

## 2. Nonpolynomial coefficients:

$$y'' + (\cos x)y = 0$$

Just like in the previous problem, since there are no singular values for  $\cos x$ , no matter what value of  $x_0$  we choose, it will converge for  $x \in \mathfrak{R}$ . So we will choose  $x_0 = 0$  for our power series.

$$y = \sum_{k=0}^{\infty} a_k x^k, \quad y'' = \sum_{k=2}^{\infty} a_k(k)(k-1)x^{k-2}, \quad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{24} \dots$$

Substituting into our equation we get:

$$\left[ 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 \dots \right] + \left[ 1 - \frac{x^2}{2} + \frac{x^4}{24} \dots \right] \left[ a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 \dots \right] = 0$$

Multiplying and simplifying, we get:

$$(2a_2 + a_0) + (6a_3 + a_1)x + \left( 12a_4 - \frac{a_0}{2} + a_2 \right) x^2 + \left( 20a_5 - \frac{a_1}{2} + a_3 \right) x^3 + \left( 30a_6 + \frac{a_0}{24} - \frac{a_2}{2} + a_4 \right) x^4 \dots = 0$$

$$2a_2 + a_0 = 0 \rightarrow a_2 = -\frac{a_0}{2}$$

$$6a_3 + a_1 = 0 \rightarrow a_3 = -\frac{a_1}{6}$$

$$12a_4 - \frac{a_0}{2} + a_2 = 0 \rightarrow a_4 = \frac{a_0}{12}$$

$$20a_5 - \frac{a_1}{2} + a_3 = 0 \rightarrow a_5 = \frac{a_1}{30}$$

$$30a_6 + \frac{a_0}{24} - \frac{a_2}{2} + a_4 \rightarrow a_6 = -\frac{a_0}{80}$$

and so on

As we saw in the last problem, we once again get a solution of the form  $y = a_0 \left( \sum_1 \right) + a_1 \left( \sum_2 \right)$ , where the two power series are the two linearly independent solutions to the homogeneous equation.