MATH 20D – Dr. Xiao
Midterm 2 Review
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General Tips

- Make sure you know when you will take the final:
  - Check http://www.math.ucsd.edu/~m3xiao/math20d/Announcements.html for timing and details
  - Not cumulative, so you need to really know the stuff from the past few weeks
- You don't want to lose points because you take an integral, derivative, Laplace transform, or partial fraction decomposition incorrectly
  - Include tables of common identities in your notes
  - Remember: when we have a expression like $\dfrac{e^{-s}+s}{(s+1)(s+2)}$, we separate the exponential parts and then factor it out before doing any partial fraction decomposition.
    - $\dfrac{e^{-s}+s}{(s+1)(s+2)} = e^{-s}\left[\dfrac{1}{(s+1)(s+2)}\right] + \dfrac{s}{(s+1)(s+2)}$
- Check your work!
  - Whenever you get a solution to a differential equation, it is easy to confirm that your solution is correct by plugging it back into the original equation

Advanced Laplace Transforms

- See Midterm 2 Review for general discussion about Laplace transforms and a table (copied from the textbook) of most common ones
- The only new things are convolutions and Dirac delta functions
- Convolutions: $f \circledast g = \int_0^t f(t-\tau)g(\tau)d\tau$, $f \circledast g = g \circledast f$
  - Relation to Laplace transform: $\mathscr{L}\{f \circledast g\} = F(s)G(s)$ and $\mathscr{L}^{-1}\{F(s)G(s)\} = f \circledast g$
  - So we can solve integro-differential equations by making sure the integral is in the format above and then taking the Laplace transform of both sides
- Dirac delta function: $\delta(x-a) = \begin{cases} \infty & x = a \\ 0 & x \neq a \end{cases}$
  - Relation to Laplace transform: $\mathscr{L}\{\delta(x-a)\} = e^{-as}$
  - Remember: $\mathscr{L}^{-1}\left\{e^{-as}F(s)\right\} = f(x-a)\mathscr{U}(x-a)$
- Example Problem 1 (Convolutions): $y(t) = te^t + \int_0^t \tau y(t-\tau)\, d\tau$

  - $Y(s) = \dfrac{1}{(s-1)^2} + \dfrac{1}{s^2}Y(s)$
  - $Y(s)\left[\dfrac{s^2-1}{s^2}\right] = \dfrac{1}{(s-1)^2}$

- $Y(s) = \dfrac{s^2}{(s-1)^3(s+1)} = \dfrac{A}{s-1} + \dfrac{B}{(s-1)^2} + \dfrac{C}{(s-1)^3} + \dfrac{D}{s+1}$
- $Y(s) = \dfrac{1}{8}\left(\dfrac{1}{s-1}\right) + \dfrac{3}{4}\left(\dfrac{1}{(s-1)^2}\right) + \dfrac{1}{2}\left(\dfrac{1}{(s-1)^3}\right) - \dfrac{1}{8}\left(\dfrac{1}{s+1}\right)$
- $y = \dfrac{1}{8}e^t + \dfrac{3}{4}te^t + \dfrac{1}{2}\left(\dfrac{1}{2}t^2e^t\right) - \dfrac{1}{8}e^{-t} = \dfrac{e^t}{8}\left[2t^2 + 6t + 1\right] - \dfrac{e^{-t}}{8}$

- Example problem 2 (Dirac-delta function): $y'' = \delta(t-2),\ y(0) = 0,\ y'(0) = 0$
  - $s^2Y(s) = e^{-2s}$
  - $Y(s) = \dfrac{e^{-2s}}{s^2} = e^{-2s}\left[\dfrac{1}{s^2}\right]$
  - $y = (t-2)\mathscr{U}(t-2)$

Series Solutions

- Because of Taylor's theorem, the power series $f(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ can be used to solve any linear differential equation
- Method of solution:
  - First put equation in standard form: $y'' + P(x)y' + Q(x)y = g(x)$ (or equivalent for higher derivatives of y)
  - Find a point x_0 where P(x), Q(x), and g(x) are all defined and have a continuous derivative
  - Set $y = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ and calculate all needed derivatives of y using the power rule
  - Find power series representations of P(x), Q(x), and g(x)
  - Plug everything into the original equation and use algebraic techniques to combine all terms into one summation: $\sum_{n=0}^{\infty} c_n(x-x_0)^n = 0$
  - Every constant c is a function of the constants a in the power series, and every value $c_n = 0,\ n \geqslant 0$ so we can solve for the values of a in our power series
    * We see that our initial conditions give us $y(x_0) = a_0,\ y'(x_0) = a_1$
- Example problem: $y'' + xy' + 2y = 0,\ y(0) = 3,\ y'(0) = -2$

$$y = \sum_{k=0}^{\infty} a_k x^k \to y' = \sum_{k=1}^{\infty} k a_k x^{k-1} \to y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

$$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + x \sum_{k=1}^{\infty} k a_k x^{k-1} + 2 \sum_{k=0}^{\infty} a_k x^k = 0$$

– $$\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} 2 a_k x^k = 0$$

$$2a_2 + 2a_0 + \left[\sum_{k=3}^{\infty} k(k-1) a_k x^{k-2} + \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=1}^{\infty} 2 a_k x^k \right] = 0$$

$a_0 = 3$
$a_1 = -2$
$a_2 = -3$
$(k+2)(k+1)a_{k+2} + (k+2)a_k = 0,\ k \geqslant 1$

– $a_n = -\dfrac{a_{n-2}}{n-1},\ n \geqslant 3$
$a_3 = 1$
$a_4 = 1$
$a_5 = -\dfrac{1}{4}$
$\vdots$

– $y = 3 - 2x - 3x^2 + x^3 + x^4 - \dfrac{x^5}{4} + \ldots$

Systems of First-Order Differential Equations

- We might have a system of equations like this: $\frac{d}{dt}\mathbf{x}(t) = A\mathbf{x}(t) + \mathbf{g}(t)$, where $\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \end{bmatrix}$ and A is a square matrix with real (constant) values in all entries
  - Remember, this is just a fancier way of writing a system of equations
- Method of solution:
  - First solve homogeneous equation $\mathbf{x}' = A\mathbf{x}$
    * Solutions will be of the form $\mathbf{x} = e^{rt}\mathbf{u}$, where r is an eigenvalue of the system and u is a corresponding eigenvector
      · Eigenvalue equation is $(A - r\mathbf{I})\mathbf{u} = \mathbf{0}$
      · Take the determinant of $A - r\mathbf{I}$ and set equal to zero to solve for the eigenvalues
      · Determinant of a 2x2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $ad - bc$

        - Determinant of a 3x3 matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$: $a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$, where the straight lines indicate determinants of the 2x2 matrices
        - Distinct real eigenvalues: solution is $\mathbf{x} = c_1\mathbf{u}_1e^{r_1t} + c_2\mathbf{u}_2e^{r_2t}$
        - Repeated eigenvalue: solution is $\mathbf{x} = c_1\mathbf{u}e^{rt} + c_2e^{rt}[t\mathbf{u} + \mathbf{v}]$ where $(A - r\mathbf{I})\mathbf{v} = \mathbf{u}$
        - Imaginary eigenvalues: solution is $\mathbf{x} = c_1e^{\alpha t}[\mathbf{a}\cos\beta t - \mathbf{b}\sin\beta t] + c_2e^{\alpha t}[\mathbf{a}\sin\beta t + \mathbf{b}\cos\beta t]$, where the eigenvalues are $\alpha \pm \beta i$ and eigenvectors are $\mathbf{u} = \mathbf{a} \pm \mathbf{b}i$
        - If the system has 3 variables instead of 2, then we will have 3 fundamental solutions (but same format as described above)
    * Then put solutions into form $\mathbf{x} = \mathbf{Xc}$, where $\mathbf{X}$ stores the fundamental solutions and $\mathbf{c}$ is a column vector containing the constants
  – Now to solve the nonhomogeneous we will have solutions of form $\mathbf{x} = \mathbf{Xc} + \mathbf{x}_p$, just need to determine $\mathbf{x}_p$:
    * Undetermined coefficients: we will follow very similar rules for test functions as we did previously with this method, then solve for the values of each vector
    * Variation of parameters: the textbook shows a nice derivation of this formula, which is pretty similar to the previous derivation for our second order equations, and the result is that $\mathbf{x}_p = \mathbf{X}\int \mathbf{X}^{-1}\mathbf{g}\,dt$, where $\mathbf{X}^{-1}$ is the inverse of $\mathbf{X}$ ($\mathbf{X}^{-1}\mathbf{X} = \mathbf{X}\mathbf{X}^{-1} = \mathbf{I}$)
        - For a 2x2 matrix, we can calculate $\mathbf{X}^{-1}$ directly: $\mathbf{X}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, for $\mathbf{X} = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}$

- Example problem 1 (Undetermined Coefficients): $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -7 \\ 5 \end{bmatrix}$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2-r & 3 \\ -1 & -2-r \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{0}$$

$$(2-r)(-2-r)-(-1)(3) = 0$$
$$r^2 - 1 = 0 \rightarrow r = \pm 1$$

$r = 1:$
$$\begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{0}$$
$$u_1 + 3u_2 = 0$$
$$u_2 = c_1 \rightarrow u_1 = -3c_1$$

- $\rightarrow c_1 e^t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$r = -1:$
$$\begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{0}$$
$$u_1 = -u_2$$
$$u_2 = c_2 \rightarrow u_1 = -c_2$$
$$\rightarrow c_2 e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\mathbf{x}_p = \begin{bmatrix} a \\ b \end{bmatrix} \rightarrow \mathbf{x}_p{}' = \mathbf{0}$$
$$\mathbf{0} = \begin{bmatrix} 2a+3b-7 \\ -a-2b+5 \end{bmatrix}$$

- $2a + 3b = 7$
  $a + 2b = 5$
  $b = 3 \rightarrow a = -1$

$$\mathbf{x}_p = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

- $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

- Example problem 2 (Variation of Parameters): $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_h = \begin{bmatrix} -3e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} -3e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix}^{-1} = \begin{bmatrix} -0.5e^{-t} & -0.5e^{-t} \\ 0.5e^t & 1.5e^t \end{bmatrix}$$

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$$\mathbf{x}_p = \begin{bmatrix} -3e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix} \int \begin{bmatrix} -0.5e^{-t} & -0.5e^{-t} \\ 0.5e^t & 1.5e^t \end{bmatrix} \begin{bmatrix} e^t \\ e^{-t} \end{bmatrix} dt$$

$$= \begin{bmatrix} -3e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix} \begin{bmatrix} -0.5t + 0.25e^{-2t} \\ 0.25e^{2t} + 1.5t \end{bmatrix} = \begin{bmatrix} -3e^t\left(-0.5t + 0.25e^{-2t}\right) - e^{-t}\left(0.25e^{2t} + 1.5t\right) \\ e^t\left(-0.5t + 0.25e^{-2t}\right) + e^{-t}\left(0.25e^{2t} + 1.5t\right) \end{bmatrix}$$

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$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \frac{1}{4}\left(2te^t \begin{bmatrix} 3 \\ -1 \end{bmatrix} + e^{-t} \begin{bmatrix} -3 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 6te^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right)$$