MATH 20D – Dr. Xiao
Midterm 1 Review
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Separable Differential Equations

- Formula
  - $y' = \dfrac{a(x)}{b(y)}$
- Method of Solution
  - First check if $y' = 0$, solve for y and see if this equation is a solution
  - Then we can rearrange the equation to: $b(y)dy = a(x)dx$ and integrate
- Example
  - Population P(t) can be described as $P' = (k\cos t)P$. Find the solution if the initial population is 100, and k = 5
    - $\displaystyle\int \frac{1}{P}dP = \int (k\cos t)dt$
    - $P = Ce^{k\sin t}$
    - $P = 100e^{5\sin t}$

Linear First-Order Equations

- Formula
  - $y' + P(x)y = Q(x)$
- Method of Solution
  - Calculate integrating factor $\mu(x) = e^{\int P(x)dx}$
  - $y = \dfrac{\int \mu(x)Q(x)dx \ + \ C}{\mu(x)}$
- Example
  - Solve $y' + 3x^2y = x^2$ with initial condition y(0) = 2.
  - $\mu(x) = e^{x^3}$
  - $y = \dfrac{\int e^{x^3}x^2dx}{e^{x^3}} \to u = x^3 \to du/3 = x^2dx \to y = \dfrac{\int e^u du}{3e^{x^3}} = \dfrac{1}{3} + \dfrac{c}{3e^{x^3}}$
  - $2 = \dfrac{1}{3} + \dfrac{c}{3e^0} = \dfrac{1}{3} + \dfrac{c}{3} \to c = 5$

Exact Equations

- Formula
  - $M(x,y) + N(x,y)\dfrac{dy}{dx} = 0$
  - We must either show $\displaystyle\int M(x,y)dx = \int N(x,y)dy = F(x,y)$ or

$$\frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y) = \frac{\partial^2 F(x,y)}{\partial x\,\partial y}$$

- Method of Solution
  - Integrate M and N in terms of x and y, respectively, and solve for F.
  - $F(x,y) + c = 0$ is an implicit solution
- Example
  - $M = (x+y)^2;\ N = \left(2xy + x^2 + y^2\right)$
  - $\int M dx = \frac{(x+y)^3}{3} + g(y) = \frac{x^3}{3} + x^2y + xy^2 + \frac{y^3}{3} + g(y)$
    $\int N dy = xy^2 + x^2y + \frac{y^3}{3} + h(x)$
  - $h(x) = \frac{x^3}{3} + c,\ g(y) = c$
  - $F(x,y) = \frac{(x+y)^3}{3} + c = 0$
  - $c = -\frac{8}{3}$

Special Integrating Factors

- Formula:
  - $M(x,y) + N(x,y)\,y' = 0$
  - We must either show:
    * Case 1: $\frac{M_y - N_x}{N}$ is solely a function of x, *or*
    * Case 2: $\frac{N_x - M_y}{M}$ is solely a function of y
- Method of solution:
  - Multiply both sides of equation by integrating factor $\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$ (Case 1) or $\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$ (Case 2)
  - The equation is now exact and we can solve it as described previously
- Example:
  - Solve $x + \left(x^2y + 4y\right)y' = 0,\ y(4) = 0$
  - $M = x,\ N = x^2y + 4y \rightarrow M_y = 0,\ N_x = 2xy$
  - $\frac{M_y - N_x}{N} = \frac{-2x}{\left(x^2+4\right)}, \frac{N_x - M_y}{M} = \ 2y$
  - Multiply by $\mu(y) = e^{y^2}$ on both sides, so new equation is $xe^{y^2} + \left(x^2y + 4y\right)e^{y^2}y' = 0$
  - $M_{exact} = xe^{y^2},\ N_{exact} = \left(x^2 + 4\right)ye^{y^2}$

$$\int xe^{y^2}dx = \frac{x^2e^{y^2}}{2} + g(y)$$

– $\int (x^2+4)ye^{y^2}dy \rightarrow u = y^2 \rightarrow \frac{du}{2} = ydy \rightarrow \int (x^2+4)e^u du =$

$(x^2+4)\frac{e^{y^2}}{2} + h(x) = \frac{x^2e^{y^2}}{2} + 2e^{y^2} + h(x)$

– $g(y) = 2e^{y^2} + c,\ h(x) = c$

– $F(x,y) = \frac{x^2e^{y^2}}{2} + 2e^{y^2} + c = 0$

– $c = -10$

Homogeneous Linear Second-Order ODEs with Constant Coefficients

- Equation:
- Method of solution:
  - Assuming that this equation will have a solution of the form $y = e^{mx}$
  - Solve the *characteristic equation*: $am^2 + bm + c = 0$ using quadratic formula:

    $m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
  - Three cases for our solution:
    * Two real roots: general solution is $y = c_1e^{m_1x} + c_2e^{m_2x}$
    * One real root: general solution is $y = c_1e^{mx} + c_2xe^{mx}$
    * Imaginary roots: general solution is $y = e^{\alpha x}[c_1\cos\beta x + c_2\sin\beta x]$, where $m = \alpha \pm \beta i$
- Example:
  - Solve $3y'' + 2y' + y = 0,\ y(0) = 5,\ y'(0) = 1$
  - $3m^2 + 2m + 1 = 0 \rightarrow m = -\frac{1}{3} \pm i\frac{\sqrt{2}}{3}$
  - $y = e^{-x/3}\left[c_1\cos\left(\frac{\sqrt{2}}{3}x\right) + c_2\sin\left(\frac{\sqrt{2}}{3}x\right)\right]$
  - After taking derivative and substituting for inital conditions we get $c_1 = 5,\ c_2 = 4\sqrt{2}$