MATH 20D – Dr. Xiao
Midterm 2 Review
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General Tips

- You don't want to lose points because you take an integral or a derivative incorrectly!
  - Include in your notes a table of common integrals and derivatives
  - Review the product rule and chain rule for derivatives
  - Review u-substitution and integration by parts for integrals
- You don't want to lose points because of a sign error!
  - Very common on the last midterm
  - Many of these are easily avoidable by writing your work more neatly and being meticulous about the steps you take (this also makes TAs happy!)
- Check your work!
  - Whenever you get a solution to a differential equation, it is easy to confirm that your solution is correct by plugging it back into the original equation

Cauchy-Euler Equations

- Equation:
  - $ax^2y'' + bxy' + cy = 0, a \neq 0$
- Method of solution:
  - Find solutions of the form $y = x^r$
  - $ar^2 + (b - a)r + c = 0$ is our *characteristic equation*
  - Use quadratic formula to solve for r and don't forget about imaginary roots
  - Three cases for our solution:
    * Two real roots: general solution is $y = c_1x^{r_1} + c_2x^{r_2}$
    * One real root: general solution is $y = c_1x^r + c_2x^r \ln x$
    * Imaginary roots:
      · These will be of form $r = \alpha \pm \beta i$
      · General solution is $y = x^\alpha[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$
- Example Problem: $x^2y'' + 8xy' + 6y = 0$
  - $r^2 + (8 - 1)r + 6 = 0 \rightarrow (r + 1)(r + 6) = 0 \rightarrow r = -1, -6$
  - $y = c_1x^{-1} + c_2x^{-6}$

Nonhomogeneous Equations

- Already know how to solve constant-coefficient and Cauchy-Euler homogeneous equations, which give us general solution $y_h = c_1y_1 + c_2y_2$
- Superposition principle: general solution to a nonhomogeneous equation is $y = y_h + y_p$ for y_h general solution to corresponding homogeneous problem and y_p a particular solution we must find
- Method of solution is different depending on what the nonhomogeneous function is:
  - $y'' + y' + 1 = g(x)$

- We can use these test functions to solve if the function is the product of powers of t, e^t, and sine or cosine of t:

Method of Undetermined Coefficients

To find a particular solution to the differential equation

$$ay''+by'- cy = Ct^m e^{rt},$$

where *m* is a nonnegative integer, use the form

$$y_p(t) = t^s (A_m t^m + \cdots + A_1 t + A_0) e^{rt},$$

with

i. $s = 0$ if *r* is not a root of the associated auxiliary equation;
ii. $s = 1$ if *r* is a simple root of the associated auxiliary equation; and
iii. $s = 2$ if *r* is a double root of the associated auxiliary equation.

To find a particular solution to the differential equation

$$ay''+by'+cy = \begin{cases} Ct^m e^{\alpha t} \cos \beta t \\ \text{or} \\ Ct^m e^{\alpha t} \sin \beta t \end{cases}$$

for $\beta \neq 0$, use the form

$$\begin{aligned} y_p(t) &= t^s (A_m t^m + \cdots + A_1 t + A_0) e^{\alpha t} \cos \beta t \\ &+ t^s (B_m t^m + \cdots + B_1 t + B_0) e^{\alpha t} \sin \beta t, \end{aligned}$$

with

iv. $s = 0$ if $\alpha + i\beta$ is not a root of the associated auxiliary equation; and
v. $s = 1$ if $\alpha + i\beta$ is a root of the associated auxiliary equation.

- Otherwise, we must use variation of parameters: $y_p = u_1 y_1 + u_2 y_2$
  * $u_1 = -\int \frac{y_2 g(x)}{a(x)[y_1 y_2' - y_1' y_2]} dx$ and $u_2 = \int \frac{y_1 g(x)}{a(x)[y_1 y_2' - y_1' y_2]} dx$, where $a(x) = a$ for constant-coefficient problems or $a(x) = ax^2$ for Cauchy-Euler problems

- Example Problem – Undetermined Coefficients: $y'' + 3y = -48x^2 e^{3x}$
  - $y_h'' + 3y_h = 0 \rightarrow m^2 + 3 = 0 \rightarrow m = \pm i\sqrt{3} \rightarrow y_h = c_1 \sin\sqrt{3}x + c_2 \cos\sqrt{3}x$
  - $y_p = \left(A + Bx + Cx^2\right)e^{3x} \rightarrow y_p' = \left(B + 3A + (2C + 3B)x + 3Cx^2\right)e^{3x}$ $\rightarrow y_p'' = (2C + 6B + 9A) + (12C + 9B)x + 9Cx^2)e^{3x}$
  - $(2C + 6B + 12A) + (12C + 12B)x + 12Cx^2 = -48x^2 + 0x + 0$
  - $\begin{cases} x^2: & 12C = -48 \rightarrow C = -4 \\ x^1: & 12C + 12B = 0 \rightarrow B = 4 \\ x^0: & 2C + 6B + 12A = 0 \rightarrow A = -\frac{4}{3} \end{cases}$
  - $y = c_1 \sin\sqrt{3}x + c_2 \cos\sqrt{3}x + \left(-\frac{4}{3} + 4x - 4x^2\right)e^{3x}$

- Example Problem – Variation of Parameters: $y'' + 3y' + 2y = \sin\left(e^x\right)$

- $y_h'' + 3y_h' + 2y_h = 0 \rightarrow m^2 + 3m + 2 = 0 \rightarrow m = -1, -2$
- $y_h = c_1 e^{-x} + c_2 e^{-2x}$
- $y_p = u_1 y_1 + u_2 y_2$
- $u_1 = \int e^x \sin(e^x) dx \rightarrow u = e^x,\ u' = e^x \rightarrow \int \sin u\, du = -\cos(e^x)$
- $u_2 = -\int e^{2x} \sin(e^x) dx \rightarrow u = e^x \rightarrow -\int u \sin(u)\, du \rightarrow e^x \cos(e^x) - \sin(e^x)$
- $y_p = -e^{-2x} \sin(e^x)$
- $y = c_1 e^{-x} + c_2 e^{-2x} - e^{-2x} \sin(e^x)$

Laplace Transforms

- We can solve many linear initial value problems using the method of Laplace transforms:

Method of Laplace Transforms

To solve an initial value problem:

a. Take the Laplace transform of both sides of the equation.
b. Use the properties of the Laplace transform and the initial conditions to obtain an equation for the Laplace transform of the solution and then solve this equation for the transform.
c. Determine the inverse Laplace transform of the solution by looking it up in a table or by using a suitable method (such as partial fractions) in combination with the table.

- Caveats:
  - Sometimes our initial conditions aren't sufficient to find a unique solution
  - Sometimes we don't know the Laplace transform of a function
  - Sometimes the other methods we have already learned are faster
- Definition: $\mathscr{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x) dx$
- Linearity property of Laplace and inverse transforms:
  - $\mathscr{L}\{af(x) + bg(x)\} = a\mathscr{L}\{f(x)\} + b\mathscr{L}\{g(x)\}$
  - $\mathscr{L}^{-1}\{aF(s) + bG(s)\} = a\mathscr{L}^{-1}\{F(s)\} + b\mathscr{L}^{-1}\{G(s)\}$
- All transforms listed in textbook (page IEP6):

A Table of Laplace Transforms

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$ $f(t)\ F(s) = \mathscr{L}\{f\}(s)$
1. $f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
2. $e^{at} f(t)$	$F(s-a)$
3. $f\prime(t)$	$sF(s) - f(0)$
4. $f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f\prime(0)$ $- \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$
5. $t^n f(t)$	$(-1)^n F^{(n)}(s)$
6. $\frac{1}{t} f(t)$	$\int_s^\infty F(u)du$
7. $\int_0^t f(v)dv$	$\frac{F(s)}{s}$
8. $(f^*g)(t)$	$F(s)G(s)$
9. $f(t+T) = f(t)$	$\frac{\int_0^T e^{-st} f(t)dt}{1-e^{-sT}}$
10. $f(t-a)u(t-a),\ a \geq 0$	$e^{-as} F(s)$
11. $g(t)u(t-a),\quad a \geq 0$	$e^{-as} \mathscr{L}\{g(t+a)\}(s)$
12. $u(t-a),\quad a \geq 0$	$\frac{e^{-as}}{s}$
13. $\prod_{a,b}(t),\quad 0 < a < b$	$\frac{e^{-sa} - e^{-sb}}{s}$
14. $\delta(t-a),\quad a \geq 0$	e^{-as}
15. e^{at}	$\frac{1}{s-a}$
16. $t^n,\quad n = 1, 2, \ldots$	$\frac{n!}{s^{n+1}}$
17. $e^{at}t^n,\quad n = 1, 2, \ldots$	$\frac{n!}{(s-a)^{n+1}}$
18. $e^{at} - e^{bt}$	$\frac{(a-b)}{(s-a)(s-b)}$
19. $ae^{at} - be^{bt}$	$\frac{(a-b)s}{(s-a)(s-b)}$
20. $\frac{1}{\sqrt{t}}$	$\frac{\sqrt{\pi}}{\sqrt{s}}$
21. $\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$
22. $t^{n-(1/2)},\quad n = 1, 2, \ldots$	$\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}}{2^n s^{n+(1/2)}}$
23. $t^r,\quad r > -1$	$\frac{\Gamma(r+1)}{s^{r+1}}$
24. $\sin bt$	$\frac{b}{s^2+b^2}$
25. $\cos bt$	$\frac{s}{s^2+b^2}$
26. $e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$
27. $e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
28. $\sinh bt$	$\frac{b}{s^2-b^2}$
29. $\cosh bt$	$\frac{s}{s^2-b^2}$
30. $\sin bt - bt \cos bt$	$\frac{2b^3}{(s^2+b^2)^2}$
31. $t \sin bt$	$\frac{2bs}{(s^2+b^2)^2}$
32. $\sin bt + bt \cos bt$	$\frac{2bs^2}{(s^2+b^2)^2}$
33. $t \cos bt$	$\frac{s^2-b^2}{(s^2+b^2)^2}$
34. $\sin bt \cosh bt - \cos bt \sinh bt$	$\frac{4b^3}{s^4+4b^4}$
35. $\sin bt \sinh bt$	$\frac{2b^2 s}{s^4+4b^4}$
36. $\sinh bt - \sin bt$	$\frac{2b^3}{s^4-b^4}$
37. $\cosh bt - \cos bt$	$\frac{2b^2 s}{s^4-b^4}$
38. $J_v(bt),\ v > -1$	$\frac{\left(\sqrt{s^2+b^2}-s\right)^v}{b^v \sqrt{s^2+b^2}}$

- Usually we will have a fraction of the form $Y(s) = \dfrac{P(s)}{Q(s)}$ and we need to use partial fraction decomposition to split this into fractions which we can individually take the inverses of
- Example Problem: $y'' + 6y' + 5y = x - x\mathscr{U}(x-2),\ y(0) = 1,\ y'(0) = 0$

$$\mathscr{L}\{y'' + 6y' + 5y\} = \mathscr{L}\{x - x\mathscr{U}(x-2)\}$$

– $s^2Y(s) - s + 6(sY(s) - 1) + 5Y(s) = \dfrac{1}{s^2} - e^{-2s}\left[\dfrac{1}{s^2} + \dfrac{2}{s}\right]$

- $Y(s) = \dfrac{s^3 + 6s^2 + 1}{s^2(s+5)(s+1)} - e^{-2s}\left[\dfrac{2s+1}{s^2(s+5)(s+1)}\right]$
- $Y(s) = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s+1} + \dfrac{D}{s+5} - e^{-2s}\left[\dfrac{E}{s} + \dfrac{F}{s^2} + \dfrac{G}{s+1} + \dfrac{H}{s+5}\right]$
- $\dfrac{s^3 + 6s^2 + 1}{s^2(s+5)(s+1)} = \dfrac{-6/25}{s} + \dfrac{1/5}{s^2} + \dfrac{3/2}{s+1} + \dfrac{-13/50}{s+5}$
- $\dfrac{2s+1}{s^2(s+5)(s+1)} = \dfrac{4/25}{s} + \dfrac{1/5}{s^2} + \dfrac{-1/4}{s+1} + \dfrac{9/100}{s+5}$
- $y(x) = -\dfrac{6}{25} + \dfrac{x}{5} + \dfrac{3e^{-x}}{2} - \dfrac{13e^{-5x}}{50} - \mathscr{U}(x-2)\left[\dfrac{4}{25} + \dfrac{x-2}{5} - \dfrac{e^{-(x-2)}}{4} + \dfrac{9e^{-5(x-2)}}{100}\right]$