Lecture 1

§ 1.1 – 1.2

Q: What is a Differential Equation (D.E)?

E.g ① $\frac{dy}{dx} = -2y$

② $\frac{d^2u}{dt^2} + \frac{du}{dt} + t = 0$

Def^n: An equation that contains derivatives of some unknown function is called a differential equation.

Q: Why do we study D.E?

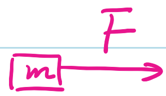
A: D.Es are not only important in Math, but also has wide applications in Science and engineering, etc.

physics

E.g. "Newton's Second Law" is a D.E.

$$F = ma$$

external force / mass / acceleration



$$a = \frac{dv}{dt} = \frac{d^2 s}{dt^2}$$

v: velocity

s: position

E.g. Biology

Exponential growth model

Let P: the population of some bacteria on a petri dish.

Assume there is enough food and space.

Then the exponential growth model says

$$\frac{dP}{dt} = kP.$$

$k>0$ constant

Q: How do we study D.Es (in this course)?

A:

- For "easy" D.Es, we explicitly solve them.

- Most D.Es are so difficult that we cannot find explicit solutions!

In lots of cases, we can still say

something about the D.E:

e.g. ① Does D.E have a soln?

② Is the soln unique if exists?

Sometimes, people use computational ways to find approximate solutions.

The main objective of 2OD is to study D.Es by using Calculus theorem.

How to identify these easy D.Es that we can explicitly solve?

For that, we first introduce some definitions for D.Es.

<u>Def</u>

Def^n: A D.E always involves the derivative of some variable(s) with respect to some other variable(s).

The former: dependent variable(s)

The latter: independent variable(s)

E.g ① $\frac{dx}{dt} = x^2 + \cos t$ 1st order ODE

x: dep. variable
→ t: indep. variable

2nd order PDE

② $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 1$$

u: dep. variable

$\left\{\begin{array}{l} x: \\ y: \end{array}\right.$ indep. variables

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Def^n: We call

- A D.E that has only one independent variable — ordinary differential equation (ODE)

- A D.E that has 2 or more independent variables — partial differential equation (PDE)

Def^n : The order of a D.E

= the order of the highest derivatives present in the eqn.

E.g

$$\frac{d^5y}{dx^5} + \frac{d^2y}{dx^2} + y^2 + x = 0$$

5th order D.E

Linear D.E v.s Non-Linear D.E

Def^n: ① A D.E is call linear if it has the format:

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)\underset{\uparrow\ \frac{d^0 y}{dx^0}}{y} = F(x)$$

The coefficients: $a_n, \cdots, a_1, a_0$
RHS: F
} only depend on x

② If a D.E is NOT linear, then we say it is a non-linear D.E.

E.g: ① $x^2\frac{dy}{dx} + \sin x = 0$ Linear D.E

② $\frac{d^3y}{dx^3} + \frac{dy}{dx} + x^2 = 0$ Linear D.E

① $x^2\frac{dy}{dx} = -\sin x$

$$\underbrace{a_1(x)}_{x^2 =} \cdot \frac{dy}{dx} + 0 \cdot y = F(x)$$

② $$\underset{a_2}{1}\cdot\frac{d^3y}{dx^3}+\underset{a_1}{1}\cdot\frac{dy}{dx}+\underset{a_0=0}{0}\cdot y=\underbrace{-x^2}_{F(x)}$$

③ $$\underbrace{\left(\frac{dy}{dx}\right)^2}_{\frac{dy}{dx}\cdot\frac{dy}{dx}}=x^2$$ non-Linear D.E

④ $$\frac{dy}{dx}\frac{d^2y}{dx^2}+x=0$$ non-Linear D.E

⑤ $$\frac{dy}{dx}+\sin(y)=0$$

$$1\cdot\frac{dy}{dx}=-\sin(y)$$ non-linear D.E

Def^n: By an initial value problem (I.V.P) for an nth-order D.E, we mean

$$\begin{cases} F\left(x,y,\frac{dy}{dx},\cdots,\frac{d^ny}{dx^n}\right)=0 & \text{D.E} \\ y(x_0)=y_0,\ \frac{dy}{dx}(x_0)=y_1,\ \cdots,\ \frac{d^{n-1}y}{dx^{n-1}}(x_0)=y_{n-1} & \text{initial value/condition} \end{cases}$$

Here x_0 is a given pt. $y_0, y_1, \cdots, y_{n-1}$ are given.

A special case is when $n=1$:

I.V.P for 1st order D.E / 1st order I.V.P

$$\begin{cases} F(x, y, \frac{dy}{dx}) = 0 \\ y(x_0) = y_0 \end{cases}$$

Remark: The initial condition is used to specify the precise solution.

E.g. Solve 1st order I.V.P

$$\begin{cases} \frac{dy}{dx} - x^2 = 0 & \text{D.E} \\ y(0) = 1 & \text{I. condition} \end{cases}$$

A: $\frac{dy}{dx} - x^2 = 0 \Rightarrow$

$$\frac{dy}{dx} = x^2 \Rightarrow$$

$$y = \int x^2 dx = \frac{1}{3}x^3 + C$$

C = any constant

$y(0) = 1 \Rightarrow$

Letting $x = 0$

$$1 = y(0) = \frac{1}{3} \cdot 0^3 + c = c$$

$$\Rightarrow c = 1$$

Hence $y = \frac{1}{3}x^3 + 1$