Lecture 10

Plan: §4.5 The superposition principle

Consider: $ay'' + by' + cy = f(x)$

Last time: find "a particular soln"
"some soln"

Today: find "all" the solns to the above D.E

Thm1: (Superposition Principle)

Let $y_{p,1}$ be a particular solution to the D.E

$$ay'' + by' + cy = f_1(x)$$

and $y_{p,2}$ be a particular solution to the D.E

$$ay'' + by' + cy = f_2(x)$$

Then for any constants k_1 and k_2, the function $k_1 y_{p,1} + k_2 y_{p,2}$ is a particular soln to the D.E

$$ay'' + by' + cy = k_1 f_1(x) + k_2 f_2(x)$$

E.g.: ① Find a particular soln to

$$y'' - y = 1$$

② Find a particular soln to

$$y'' - y = x^2$$

③ Find a particular soln to

$$y'' - y = 2 - x^2$$

A: ① $RHS = 1 = 1 \cdot x^0 \cdot e^{0 \cdot x}$

$$\Rightarrow \begin{cases} m = 0 \\ r = 0 \end{cases}$$

Recall: Lecture 9

Consider $ay'' + by' + cy = C_0 x^m e^{rx}$

(I) If r is not a root of the char. eqn

$$a\lambda^2 + b\lambda + c = 0$$

then use the test function

$$y = (A_m x^m + A_{m-1} x^{m-1} + \cdots + A_1 x + A_0) e^{rx}$$

Note the char. eqn is $\lambda^2 - 1 = 0$

$r = 0$ is NOT a root of the char. eqn.

Thus we use the test function

$$y = A_0 e^{0 \cdot x} = A_0$$

Then $LHS = y'' - y = -A_0$

Note $RHS = 1$

$\Rightarrow$ we need $-A_0 = 1$

$\Rightarrow A_0 = -1$

Hence $y_{P,1} = -1$ is a particular soln to

$$y'' - y = 1$$

② $RHS = x^2 = 1 \cdot x^2 \cdot e^{0 \cdot x}$

$$\Rightarrow \begin{cases} m = 2 \\ r = 0 \end{cases}$$

Again note the char. eqn is

$$\lambda^2 - 1 = 0$$

and $r=0$ is NOT a root.

$\Rightarrow$ we use the test function

$$y = (A_2 x^2 + A_1 x + A_0)\cancelto{1}{e^{0 \cdot x}}$$

plug into $y'' - y = x^2$

$$\Rightarrow \text{LHS: } \underbrace{2A_2}_{y''} - (\underbrace{A_2 x^2 + A_1 x + A_0}_{y}) = x^2 \text{ : RHS}$$

$$\Rightarrow -A_2 x^2 - A_1 x + (2A_2 - A_0) = x^2$$

Compare the two sides:

x^2-term: $-A_2 x^2 = x^2$

x-term: $-A_1 x = 0 \cdot x$

Constant-term: $2A_2 - A_0 = 0$

$$\Rightarrow \begin{cases} -A_2 = 1 \\ -A_1 = 0 \\ 2A_2 - A_0 = 0 \end{cases} \Rightarrow \begin{cases} A_2 = -1 \\ A_1 = 0 \\ A_0 = -2 \end{cases}$$

$\Rightarrow$ $y_{p,2} = -x^2 - 2$ is a particular soln to $y'' - y = x^2$

RHS of 3rd D.E
$= 2 - x^2$
$= 2 \cdot 1 + (-1) \cdot x^2$

③ By ①, $y_{p,1} = -1$ is a particular soln to $y'' - y = 1$

By ②, $y_{p,2} = -x^2 - 2$ is a particular soln to $y'' - y = x^2$

By Thm 1,

"$k_1 y_{p,1} + k_2 y_{p,2}$" is a particular soln to $y'' - y = k_1 \cdot 1 + k_2 \cdot x^2$

In particular, let $k_1 = 2$, $k_2 = -1$

$\Rightarrow$ $y = \boxed{2y_{p,1} - y_{p,2}} = 2 \cdot (-1) - (-x^2 - 2) = x^2$

is a particular soln to

$$y'' - y = 2 - x^2$$

A very often used case of Thm 1 is $k_1 = 1$ and $k_2 = 1$. That is:

Corollary:

If $y_{p,1}$ is a particular soln to

$$ay'' + by' + cy = f_1(x)$$

and $y_{p,2}$ is a particular soln to

$$ay'' + by' + cy = f_2(x)$$

then $y_{p,1} + y_{p,2}$ is a particular soln

to $$ay'' + by' + cy = f_1(x) + f_2(x).$$

Q: What does the above Corollary say if $f_1 = f$, $f_2 = 0$.

A: Assume y_p is a particular soln to

$$ay'' + by' + cy = f(x). \quad (1)$$

Recall we know how to find general solns to

$$ay'' + by' + cy = 0 \quad (2)$$

(Assume y_1, y_2 are two linearly independent solns to (2). Then the general solns to (2): $c_1 y_1 + c_2 y_2$)

Then $y_p + (c_1 y_1 + c_2 y_2)$ is also

a soln to

$$ay'' + by' + cy = f(x) + 0 = f(x) \qquad (1)$$

Q: Are there any other solns to (1)?

A: No! "$y_p + c_1 y_1 + c_2 y_2$" gives all possible solns to (1)!

☆ Thm: Suppose y_p is a particular soln to

$$ay'' + by' + cy = f(x). \qquad (1)$$

Suppose y_1, y_2 are two linearly independent solns to

$$ay'' + by' + cy = 0. \qquad (2)$$

Then "$\underbrace{\text{the general solns}}_{\text{meaning "all solutions"}}$" to $\underline{(1)}$ are

"$y_p + c_1 y_1 + c_2 y_2$", $c_1, c_2 \in \mathbb{R}$

Idea of pf:

Let y_0 be any other soln to (1) (than y_p). "$ay_0'' + by_0' + cy_0 = f(x)$"

We want to show:

$$y_0 = y_p + c_1 y_1 + c_2 y_2$$

for some $c_1, c_2 \in \mathbb{R}$.

Claim: $(y_0 - y_p)$ is a soln to (2)

Why? Check $y_0 - y_p$ satisfies (2).

$$a(y_0 - y_p)'' + b(y_0 - y_p)' + c(y_0 - y_0)$$

$$= \underbrace{(a y_0'' + b y_0 + c y_0)}_{= f(x)} - \underbrace{(a y_p'' + b y_p' + c y_p)}_{= f(x)}$$

$$= 0 = \text{RHS of } (2)$$

Hence $y_0 - y_p$ is a soln of (2).

But we know every soln of (2) can be written as $c_1 y_1 + c_2 y_2$, $c_1, c_2 \in \mathbb{R}$

$$\Rightarrow \quad y_0 - y_p = c_1 y_1 + c_2 y_2$$

$$\Rightarrow \quad y_0 = y_p + c_1 y_1 + c_2 y_2$$

<u>Def^n</u>: "$ay'' + by' + cy = 0$" is called the associated homogeneous D.E of "$ay'' + by' + cy = f(x)$".

☆ Algorithm to find the general solns (that is, all solns) to

$$ay'' + by' + cy = f(x) \quad (1)$$

Step 1: Find the general solns to the associated homogeneous D.E

$$ay'' + by' + cy = 0$$

Call the solns $y_h = C_1 y_1 + C_2 y_2$

Step 2: Find a particular soln. to (1)

How? We can use Lecture 9 "undetermined coeff. method"

(next lecture will discuss another method)

Call the soln y_p

Step 3. Add up solns in step 1, 2

$$\Rightarrow \quad y_p + C_1 y_1 + C_2 y_2, \quad C_1, C_2 \in \mathbb{R}$$

E.g: ① Find the general solns to

$$y'' - y = 2 - x^2 \qquad (3)$$

② Solve the I.V.P

$$\begin{cases} y'' - y = 2 - x^2 \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$$

A: ① We follow the algorithm:

Step 1: Solve the asso. homog. D.E

$$y'' - y = 0.$$

The char. eqn. $\lambda^2 - 1 = 0$

$\Rightarrow \quad \lambda_1 = 1, \quad \lambda_2 = -1$

Hence $y_h = C_1 e^x + C_2 e^{-x}$

Step 2: Find a part. soln. to (3).

Already did this early today!

$$y_p = x^2$$

Step 3: Add them up

$$y = y_p + y_h$$

$$= x^2 + C_1 e^x + C_2 e^{-x}$$

This is the general solns to (3)

(2) Recall the general solns to (3):

$$y = x^2 + c_1 e^x + c_2 e^{-x}$$

$y(0)=1 \Rightarrow \quad 1 = 0^2 + c_1 e^0 + c_2 e^{-0}$

$x=0,\ y=1 \quad \Rightarrow \quad c_1 + c_2 = 1 \quad (1)$

Note $y' = 2x + c_1 e^x - c_2 e^{-x}$

$y'(0)=0 \Rightarrow \quad 0 = 2\cdot 0 + c_1 e^0 - c_2 e^{-0}$

$x=0$
$y'=0 \quad \Rightarrow \quad c_1 - c_2 = 0 \quad (2)$

$$\Rightarrow \begin{cases} c_1 = \frac{1}{2} \\ c_2 = \frac{1}{2} \end{cases}$$

The soln to I.V.P is

$$y = x^2 + \frac{1}{2}e^x + \frac{1}{2}e^{-x}$$